2.5 Dropping Next and Until

For expressing some properties, the $\bigcirc$ and $\mathcal{U}$ operators are not needed. For example, none of our examples of fairness, safety, and liveness conditions uses them. Let LTL$_\Diamond$ be the fragment of LTL that has the only temporal operator $\Diamond$, but neither $\bigcirc$ nor $\mathcal{U}$. Reasoning in LTL$_\Diamond$ turns out to be simpler than reasoning in LTL:

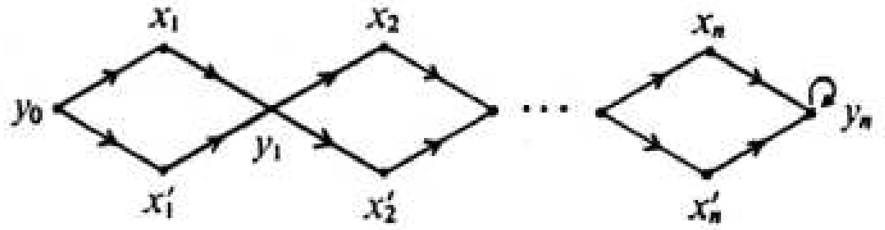
Theorem 23. *Satisfiability in LTL$_\Diamond$ is* NP*-complete, and model checking is co-*NP*-complete.*

NOTE: Not worse than propositional logic. Since propositional logic is a fragment of LTL$_\Diamond$, NP-hardness of satisfiability is immediate. We prove the following:

Lemma 24. *Model checking in LTL$_\Diamond$ is co-*NP*-hard.*

Proof. We reduce the complement of the NP-hard 3-SAT problem: given a propositional formula φ of the form $C_1 \wedge \cdots \wedge C_m$, with each C_i a disjunction of exactly three literals, decide whether φ is unsatisfiable.

Let $\varphi = C_1 \wedge \cdots \wedge C_m$ using variables $x_1, \ldots, x_n$. Let $C_i = \ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$ (for $1 \leq i \leq m$), and $\ell_{ik} = x_j$ or $\neg x_j$ for some j with $1 \leq j \leq n$ (for $1 \leq k \leq 3$). Define a Kripke structure K as follows (without valuation):



NOTE: Loop required for totality. Intuitively, each trace through K corresponds to a valuation for the variables $x_1, \ldots, x_n$. As the set of propositional letters, use $\mathsf{PL} := \{C_1, \ldots, C_m\}$. Set:

$$
\begin{array}{lcl}
V(x_i) & := & \{C_j \mid x_i \text{ appears as a literal in } C_j\} \\
V(x_i') & := & \{C_j \mid \neg x_i \text{ appears as a literal in } C_j\} \\
V(y_i) & := & \emptyset
\end{array}
$$

Then φ is satisfiable iff there is a valuation making one literal from each $C_1, \ldots, C_m$ true iff there exists a path T trough K such that $C_1, \ldots, C_m$ appear somewhere in the trace iff $T, 0 \models \Diamond C_1 \wedge \cdots \wedge \Diamond C_m$.

It follows that φ is unsatisfiable iff $K \models \neg(\Diamond C_1 \wedge \cdots \wedge \Diamond C_m)$. ❑