5. Exercises for the Course „Description Logics“

Exercise 22:
Let $C$ and $D$ be $\mathcal{ALC}$-concepts that use only the constructors $\neg$, $\cap$, and $\cup$, but not $\exists r.C$ and $\forall r.C$. With $C$, we can associate a propositional formula $\varphi_C$ by replacing each concept name $A$ with a propositional variable $p_A$, $\cup$ with $\lor$, and $\cap$ with $\land$. Similarly for $D$ and $\varphi_D$. Prove that $C \equiv D$ iff $\varphi_C \rightarrow \varphi_D$ is valid in propositional logic.

Exercise 23:
Consider the following combinations of concepts $C$ and TBoxes $T$. Determine whether $C$ is satisfiable w.r.t. $T$:
- $C = A \cap B$, $T = \{A \sqsubseteq \exists r.X \cap \exists r.\neg X, A \sqsubseteq \forall r.Y, B \equiv (\leq 1 r.Y)\}$
- $C = A \cap B$, $T = \{A \sqsubseteq (\geq 5 r.A) \cap (\leq 2 r.X), B \sqsubseteq (\leq 2 r.\neg X)\}$
- $C = A \cap \neg X$, $T = \{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.\neg X, X \sqsubseteq \forall r.(\leq 1 r.\top)\}$

Exercise 24:
Prove that the concept $\forall r.\bot$ cannot be expressed in $\mathcal{ALC}$, i.e., that there is no $\mathcal{ALC}$-concept that is equivalent to $\forall r.\bot$.

Exercise 25:
Prove that the concept $(\leq 1 r.\top)$ cannot be expressed in $\mathcal{ALC}$.

Proceed as follows: assume that $C$ is an $\mathcal{ALC}$-concept equivalent to $(\leq 1 r.\top)$, and take a model $\mathcal{I}$ of $C$.\footnote{We call $\mathcal{I}$ a model of $C$ if $C^\mathcal{I} \neq \emptyset$.} Construct a model $\mathcal{I}_\omega$ as follows:
- $\Delta^{\mathcal{I}_\omega} := \Delta^\mathcal{I} \times \mathbb{N}$;
- $A^{\mathcal{I}_\omega} := \{(d, i) \mid d \in A^\mathcal{I} \text{ and } i \geq 0\}$;
- $r^{\mathcal{I}_\omega} := \{((d, i), (d', i')) \mid (d, d') \in r^\mathcal{I} \text{ and } i, i' \geq 0\}$.

Prove that $\mathcal{I}_\omega$ is a model of $C$ and conclude that $C$ is not equivalent to $(\leq 1 r.\top)$. Can $(\leq 1 n.\top)$ be expressed in $\mathcal{ALC}$, for any $n \geq 0$?

Exercise 26:
Assume we want to define the concept of those students that attend all CL courses, using concept names such as Student and CL-Course and role names such as attends. Can this be expressed in $\mathcal{ALC}$? If not, propose a new concept-forming constructor that allows to express it. Define its syntax and semantics.