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2. Exercises for the Course „Complexity and Logic“

Exercise 5:

(a) Find two graphs G_1 and G_2 with nodes $u \in V_1$ and $v \in V_2$ such that there exists a simulation $\stackrel{\rightarrow}{\sim}$ with $u \stackrel{\rightarrow}{\sim} v$, but no homomorphism h with $h(u) = v$.

(b) Let G_1 be a tree and G_2 a graph, and let $u \in V_1$ and $v \in V_2$. Prove or refute the following:

- if there is a homomorphism with $h(u) = v$, then there exists a simulation $\stackrel{\rightarrow}{\sim}$ with $u \stackrel{\rightarrow}{\sim} v$.
- if there is a simulation $\stackrel{\rightarrow}{\sim}$ with $u \stackrel{\rightarrow}{\sim} v$, then there exists a homomorphism with $h(u) = v$.
- if u is the root of G_1 and there is a simulation $\stackrel{\rightarrow}{\sim}$ with $u \stackrel{\rightarrow}{\sim} v$, then there exists a homomorphism with $h(u) = v$.

(c) Let G_1 be a tree and G_2 a graph. Compare the algorithm for computing tree homomorphisms applied to G_1 and G_2 with the algorithm for computing simulations applied to G_1 and G_2.

(d) Can the polynomial-time algorithm for computing tree-homomorphisms be adapted to the case where G_1 is a graph and G_2 is a tree?

Exercise 6:

(a) Prove that, for all graphs G_1 and G_2, the class of simulations from G_1 to G_2 is closed under union: if $\stackrel{\rightarrow}{\sim}_1$ and $\stackrel{\rightarrow}{\sim}_2$ are simulations from G_1 to G_2, then $\stackrel{\rightarrow}{\sim}_1 \cup \stackrel{\rightarrow}{\sim}_2$ is also a simulation from G_1 to G_2.

(b) Use (a) to prove that, for all finite graphs G_1 and G_2, there exists a unique greatest simulation from G_1 to G_2.

(c) Prove or refute the following:

- For all graphs G_1, G_2, the class of simulations from G_1 to G_2 is closed under intersection.
- For any two graphs G_1, G_2, there exists a simulation from G_1 to G_2.
- For any two graphs G_1, G_2 and nodes $u \in V_1$ and $v \in V_2$, there is a simulation $\stackrel{\rightarrow}{\sim}$ from G_1 to G_2 with $u \stackrel{\rightarrow}{\sim} v$.

Exercise 7:

Convert the following pairs C_1, C_2 of $\mathcal{EL}$ concepts into trees. Check the existence of homomorphisms between the trees to decide whether $C_1 \sqsubseteq C_2$.

C_1	C_2
$\exists r.(A \sqcap B)$	$\exists r.A \sqcap \exists r.B$
$\exists r.\top \sqcap \exists r.\top$	$\top \sqcap \exists r.\top$
$\exists r.\exists r.A \sqcap \exists r.\exists s.B$	$\exists r.(\exists r.A \sqcap \exists s.B)$
$\exists r.(A \sqcap B \sqcap \exists r.A \sqcap \exists r.B)$	$\exists r.(A \sqcap \exists r.A \sqcap \exists r.B)\exists r.(B \sqcap \exists r.A \sqcap \exists r.B)$

Exercise 8:

Let G_1 and G_2 be graphs. A *bisimulation* is a relation $\sim \subseteq V_1 \times V_2$ satisfying the following properties:

- $u \sim v$ implies $\ell_1(u) = \ell_2(v)$;
- $(u_1, r, u_2) \in E_1$ and $u_1 \sim v_1$ implies that there is a $v_2 \in V_2$ with $(v_1, r, v_2) \in E_2$ and $u_2 \sim v_2$;
- $(v_1, r, v_2) \in E_2$ and $u_1 \sim v_1$ implies that there is a $u_2 \in V_1$ with $(u_1, r, u_2) \in E_1$ and $u_2 \sim v_2$.

Do the following:

(a) Develop an algorithm that, given two graphs G_1, G_2 and nodes $u \in V_1$ and $v \in V_2$, determines the existence of a bisimulation $\sim$ with $u \sim v$. Make sure the algorithm runs in polynomial time.

(b) Prove or refute the following: there exists a bisimulation between two graphs G_1 and G_2 with $u \sim v$ iff there exists a simulation $\stackrel{\rightarrow}{\sim}$ (where the comparison operator λ is equality) from G_1 to G_2 with $u \stackrel{\rightarrow}{\sim} v$, and a simulation $\stackrel{\rightarrow}{\sim}$ from G_2 to G_1 with $v \stackrel{\rightarrow}{\sim} u$.