Definition
A two-way automaton with \(k\) heads is a tuple of the form \(A = (Q, \Sigma, I, \Delta, F)\), where \(Q, \Sigma, I\) and \(F\) are as in regular two-way automata, and \(\Delta \subseteq Q \times (\Sigma \cup \{\succ, \prec\})^k \times Q \times \{-1, 0, 1\}^k\) is the transition relation.

A configuration of a two-way \(k\)-head automaton \(A\) is an element of \(Q \times \mathbb{N}^k\). A run of \(A\) on a word \(w = a_1 \ldots a_n \in \Sigma^*\) is a sequence of configurations \((q_0, j_{01}, \ldots, j_{0k}) \ldots (q_m, j_{m1}, \ldots, j_{mk})\) such that

- \(q_0 \in I\), and
- \(j_{0\ell} = 0\) for all \(\ell, 1 \leq \ell \leq k\), and
- \(0 \leq j_{i\ell} \leq n + 1\) for all \(i, 0 \leq i \leq m\), and all \(\ell, 1 \leq \ell \leq k\), and
- \((q_i, (a_{j_{i1}}, \ldots, a_{j_{ik}}), (j_{i+1,1} - j_{i1}, \ldots, j_{i+1,k} - j_{ik})) \in \Delta\) for all \(i, 0 \leq i \leq m\).

A run is accepting if \(q_m \in F\).

Exercise 1
Describe the languages that are accepted by the following two-way automata with 2 heads.

a) \(A_1 = (\{q_0, q_1, q_2\}, \{a, b\}, \{q_0\}, \Delta, \{q_2\})\), where \(\Delta\) is given by the following relation (where \(X\) and \(Y\) range over all symbols from \(\Sigma \cup \{\succ, \prec\}\)).

- \(((X, Y), (1, 0))\)
- \(((X, Y), (1, 1))\)
- \(((\prec, a), (0, 0))\)
b) $A_1 = (\{q_0, q_1, q_2\}, \{a, b\}, \{q_0\}, \Delta, \{q_2\})$, where $\Delta$ is given by the following relation (where $X$ ranges over all symbols from $\Sigma \cup \{\triangleright\}$, and $Y$ ranges over all symbols from $\Sigma$).

Exercise 2

Give formal grammars for the languages from Exercise 1. What is the least Chomsky Type of these grammars?

Exercise 3

Construct two-way multihead automata that accept the following languages.

a) $L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$

b) $L_2 = \{w \overline{w} \mid w \in \{a, b\}^*\}$

Exercise 4

Let $\varphi(x, y)$ be a first-order formula with 2 free variables. Let $\psi$ be the formula $\psi(x, y) = TC[\varphi](x, y)$ and let $\psi'$ be the formula $\psi'(x, y) = DTC[\psi](x, y)$. Give a regular expression for $L(\psi'(\min, \max))$. Prove your claim.