Chapter 4

Reasoning with tableaux algorithms

We start with an algorithm for deciding consistency of an ABox without a TBox since this covers most of the inference problems introduced in Chapter 2:

- acyclic TBoxes can be eliminated by expansion

- satisfiability, subsumption, and the instance problem can be reduced to ABox consistency

The tableau-based consistency algorithm tries to generate a finite model for the input ABox $\mathcal{A}_0$:

- applies tableau rules to extend the ABox \textit{one rule per constructor}

- checks for obvious contradictions

- an ABox that is complete (no rule applies) and open (no obvious contradictions) describes a model
Tableau algorithm

$\mathcal{T}$ \hspace{10pt} GoodStudent \equiv \text{Smart} \sqcap \text{Studious}

Subsumption question: \hspace{10pt} $\exists\text{attended}.\text{Smart} \sqcap \exists\text{attended}.\text{Studious} \subseteq^? \exists\text{attended}.\text{GoodStudent}$

Reduction to satisfiability: \hspace{10pt} is the following concept unsatisfiable w.r.t. $\mathcal{T}$?

$\exists\text{attended}.\text{Smart} \sqcap \exists\text{attended}.\text{Studious} \sqcap \neg \exists\text{attended}.\text{GoodStudent}$

Reduction to consistency: \hspace{10pt} is the following ABox inconsistent w.r.t. $\mathcal{T}$?

$\{ (\exists\text{attended}.\text{Smart} \sqcap \exists\text{attended}.\text{Studious} \sqcap \neg \exists\text{attended}.\text{GoodStudent})(a) \}$

Expansion: \hspace{10pt} is the following ABox inconsistent?

$\{ (\exists\text{attended}.\text{Smart} \sqcap \exists\text{attended}.\text{Studious} \sqcap \neg \exists\text{attended}.(\text{Smart} \sqcap \text{Studious}))(a) \}$

Negation normal form: \hspace{10pt} is the following ABox inconsistent?

$\{ (\exists\text{attended}.\text{Smart} \sqcap \exists\text{attended}.\text{Studious} \sqcap \forall\text{attended}.(\neg\text{Smart} \sqcup \neg\text{Studious}))(a) \}$
Is the following ABox inconsistent?

\( \{ (\exists \text{attended}. \text{Smart} \sqcap \exists \text{attended}. \text{Studious} \sqcap \forall \text{attended}. (\neg \text{Smart} \sqcup \neg \text{Studious}))(a) \} \)

\[ \begin{align*}
\exists r. A & \sqcap \exists r. B \sqcap \forall r. (\neg A \sqcup \neg B) \\
\exists r. A, \exists r. B, \forall r. (\neg A \sqcup \neg B) \\
\end{align*} \]

The diagram shows a complete and open ABox yields a model for the input ABox and thus a counterexample to the subsumption relationship.
Tableau algorithm

Input: An $\mathcal{ALC}$-ABox $\mathcal{A}_0$

Output: “yes” if $\mathcal{A}_0$ is consistent
“no” otherwise

Preprocessing:

transform all concept descriptions in $\mathcal{A}_0$ into negation normal form (NNF)
by applying the following rules:

$\neg(C \sqcap D) \iff \neg C \sqcup \neg D$
$\neg(C \sqcup D) \iff \neg C \sqcap \neg D$
$\neg \neg C \iff C$
$\neg (\exists r.C) \iff \forall r.\neg C$
$\neg (\forall r.C) \iff \exists r.\neg C$

The NNF can be computed in polynomial time, and it does not change the semantics of the concept.
Tableau algorithm

Data structure:
finite set of ABoxes rather than a single ABox: start with \( \{ \mathcal{A}_0 \} \)

Application of tableau rules:
the rules take one ABox from the set and replace it by finitely many new ABoxes

Termination:
if no more rules apply to any ABox in the set

Answer:
“consistent” if the set contains an open ABox, i.e., an ABox not containing an obvious contradiction of the form

\[ A(a) \text{ and } \neg A(a) \]

for some individual name \( a \)

“inconsistent” if all ABoxes in the set are closed (i.e., not open)
# Tableau rules

One for every constructor (except for negation)

### The $\sqcap$-rule

**Condition:** $\mathcal{A}$ contains $(C \sqcap D)(a)$, but not both $C(a)$ and $D(a)$

**Action:** $\mathcal{A}' := \mathcal{A} \cup \{C(a), D(a)\}$

### The $\sqcup$-rule

**Condition:** $\mathcal{A}$ contains $(C \sqcup D)(a)$, but neither $C(a)$ nor $D(a)$

**Action:** $\mathcal{A}' := \mathcal{A} \cup \{C(a)\}$ and $\mathcal{A}'' := \mathcal{A} \cup \{D(a)\}$

### The $\exists$-rule

**Condition:** $\mathcal{A}$ contains $(\exists r.C)(a)$, but there is no $c$ with $\{r(a, c), C(c)\} \subseteq \mathcal{A}$

**Action:** $\mathcal{A}' := \mathcal{A} \cup \{r(a, b), C(b)\}$ where $b$ is a new individual name

### The $\forall$-rule

**Condition:** $\mathcal{A}$ contains $(\forall r.C)(a)$ and $r(a, b)$, but not $C(b)$

**Action:** $\mathcal{A}' := \mathcal{A} \cup \{C(b)\}$
Tableau algorithm is a decision procedure for consistency

Lemma 4.1
local correctness: rules preserve consistency

Lemma 4.8
termination: no infinite paths

soundness: any complete and open ABox has a model
completeness: closed ABoxes do not have a model

Lemma 4.2