Fuzzy Logic

Lecture Notes, TU Dresden

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1 Motivation

Classical logic is very well suited for speaking of objects and properties that are identifiable, distinct and clear-cut. Many properties can be characterized this way: the days of the week, the characters in a play, the parts of an instrument, etc. In a word, these properties describe sets of individuals (that satisfy them) and vice versa.

In the real world, however, it is not hard to encounter properties that cannot be characterized in such a clear-cut manner. For example, how can one define the property "tall"? One could express that every person over 1.90m. is *tall*, but then, what about someone whose height is 1.899m.? Are we ready to express that she is *not* tall? Wherever we set the threshold, this problem is unavoidable. Other examples of

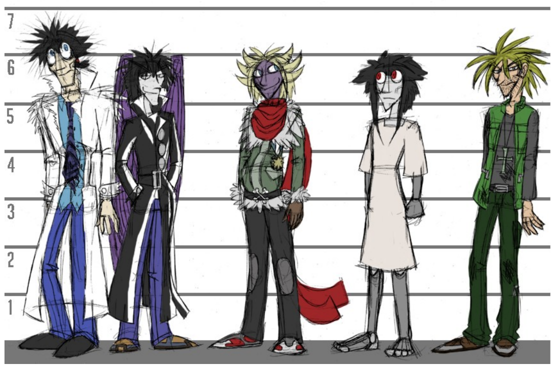

Figure 1.1: A height chart

properties that defy a crisp definition are "old" or "warm": one does not start being old from one day to the next, but rather become increasingly old with time; likewise, while a temperature of 0°C is cold and 30°C is hot, we cannot strictly express when the air is warm. A

better way to express these properties is through a *membership function* that gives a degree with which the property is satisfied. For example, two persons with height 1.90m. and 1.85m. may be tall with degree 1 and 0.8, respectively. The membership function for the property "tall"

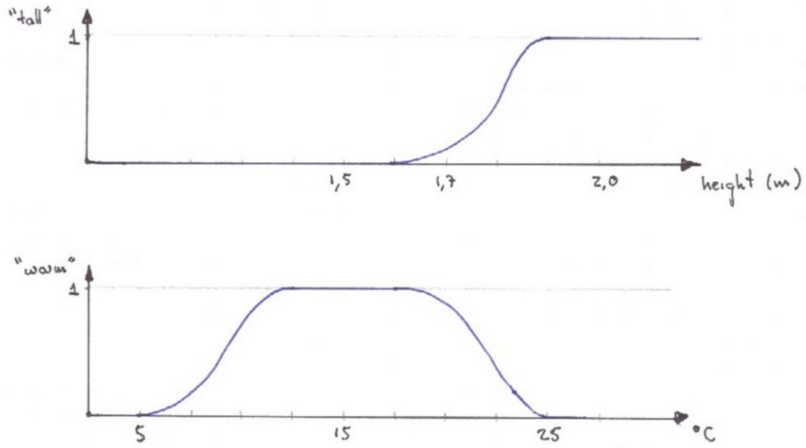

Figure 1.2: Membership functions for properties "tall" and "warm"

will be a non-decreasing function of the height of a person; that is, the greater the height, the more this person is tall. On the other hand, the membership function for "warm" will first increase with the temperature, but after some point will start decreasing again–when it stops being warm, and starts getting "hot".

We will consider the membership function to have the interval $[0, 1]$ as its image, where 0 will be interpreted as absolute false, and 1 as absolute true. The intuition is that this function generalizes the characteristic function of the sets describing a property (where an element in the set has membership 1 and all others have membership 0).

Once that such a membership function has been constructed for describing some given properties, one must be able to deduce consequences from the combinations of these properties; for example, one should be able to express to which degree is a person tall AND old. Moreover, we want to have a formal system, with unambiguous semantics, for performing these deductions. In this course, we will look at ways to generalize the classical logical operators so that they can deal with degrees of truth, rather than just the truth values 0 and 1. We will only consider operators that "behave well" logically: they should be *truth functional* (that is, the degree of a complex formula is a function of the degrees of its subformulas), conjunction should be commutative and associative, etc. This is what we call fuzzy logic. We notice that there is a wide selection of operators that "behave well", and hence the term fuzzy logic does not refer to *one* unique logic, but rather to a family of fuzzy logic systems.

There are two things to keep in mind when dealing with fuzzy logic. First, fuzzy logic deals with *vagueness*, rather than beliefs or uncertainty. The proposition "tomorrow will rain" is either (totally) true or (totally) false, we just do not know which one is the case; we can believe one or the other will happen, but we have no full certainty of it. On the other hand, two persons may be tall with a different degree, but this does not mean that it is unknown whether they are tall or not.

Second, although probabilities are also measured using the interval $[0, 1]$, fuzzy logic is *not* a probabilistic logic. For one thing, probabilities are not truth functional; for another, probabilities usually measure uncertainty or beliefs of an event, which, as said before, is outside the realm of fuzzy logic.

2 Fuzzy Propositional Logic

In this chapter we will define the basic fuzzy propositional logic. We first define the class of operators that will be the basis for the logical connectives, then introduce the basic multi-valued logic, and finally analyse some of its algebraic properties.

2.1 Continuous t-norms

Triangular norms (or t-norms for short) will be the basis under which we will be able to define the logical connectives for fuzzy logic. Before we define these formally, recall that we have chosen to take the interval $[0, 1]$ as the set of truth values, where the usual ordering $\leq$ for real numbers is translated directly to the "degree of truth". Thus, we have infinitely many truth values, organized through a linear ordering which is dense and complete. These properties will become important in the development of fuzzy logic.

Recall also that we want only to deal with connectives that are truth functional. Formally, for each binary connective c, there must exists a function $f_c : [0, 1]^2 \to [0, 1]$ such that, for any two formulas ϕ, ψ, the truth degree of the formula $c(\phi, \psi)$ is given by applying f_c to the truth degrees of ϕ and ψ. Analogously, we can deal with connectives of arbitrary arity. This requirement will be helpful for obtaining a well-behaved logic.

Finally, when choosing functions that describe a connective, we will always focus on generalizing standard two-valued logic. In other words, whenever restricted to only the truth values $0, 1$, these functions should behave as the classical connective. For example, the truth function $\otimes$ for conjunction must be such that $1 \otimes 1 = 1$ and $1 \otimes 0 = 0 \otimes 1 = 0 \otimes 0 = 0$. One obvious function that does this is the one defined by the minimum of its two arguments. It is not hard, however to define other functions satisfying these properties.

We will start the definition of fuzzy logic connectives by formulating restrictions on the truth function of conjunction. We will then show that all other connectives can be defined from conjunction in a unique and adecuate way. The truth value of conjunction will be required to be a t-norm.

Definition 2.1. A *t-norm* is an associative and commutative binary operator $\otimes$ on $[0,1]$ that is non-decreasing in both arguments and has 1 as its identity element. In other words, $\otimes$ must satisfy the following three conditions:

1. if $x_1 \leq x_2$ and $y_1 \leq y_2$, then $x_1 \otimes y_1 \leq x_2 \otimes y_2$ (non-decreasing), and

2. $1 \otimes x = x$ for all $x \in [0,1]$ (unit)

The t-norm $\otimes$ is called *continuous* if it is a continuous function (in the usual analysis sense). △

The choice of t-norms for defining the conjunction (which will, from now on, be denoted as &) is based on the following intuition. If the formula $\phi \& \psi$ has a high truth degree, then each of its subformulas ϕ and ψ must also have a high truth degree (non-decreasing) but there is no information on which one of these subformulas, if any, has a higher truth degree (commutativity). The other conditions follow from generalizing the properties of classical conjunction.

There are three important continuous t-norms that we will study during this course.

Exercise. Show that the following three binary operators are continuous t-norms:

1. *Lukasiewicz t-norm*: $x \otimes y = \max\{x + y - 1, 0\}$,

2. *Product t-norm*: $x \otimes y = x \cdot y$,

3. *Gödel t-norm*: $x \otimes y = \min\{x, y\}$.

△

We now focus on defining the implication between formulas. As before, we will define the truth function of this connective by generalizing the properties of classical implication. First, notice that the implication $\phi \rightarrow \psi$ is true in classical logic iff the truth value of ϕ is smaller or equal to the truth value of ψ. This means that the truth value of $\phi \rightarrow \psi$ decreases if the truth value of ϕ increases or the value of ψ decreases. We thus require the truth function $\Rightarrow$ of implication to be non-increasing in the first and non-decreasing in the second parameter. Additionally, we want the implication to satisfy *modus ponens*: if x is true and $x \rightarrow y$ is true, then y is also true. This generalizes to the fuzzy setting in a natural way:

$$\text{if } w \leq x \text{ and } z \leq x \Rightarrow y, \text{ then } w \otimes z \leq y,$$

and in particular (by taking $w = x$),

$$\text{if } z \leq x \Rightarrow y, \text{ then } x \otimes z \leq y.$$

In order to have the strongest possible notion of modus ponens, we then want to define $\Rightarrow$ to be as large as possible. Equivalently, we want the previous necessary condition for z to also be sufficient.

The following lemma shows that all these conditions can be satisfied in a unique way for every continuous t-norm.

Lemma 2.2. *For every continuous t-norm $\otimes$, there is a unique binary operator $\Rightarrow$ such that for every $x, y, z \in [0,1]$ it holds:*

$$z \leq x \Rightarrow y \text{ iff } x \otimes z \leq y.$$

This operator is defined as $x \Rightarrow y := \max\{z \mid x \otimes z \leq y\}$.

Proof. Let $x, y \in [0,1]$ and define $(x \Rightarrow y) = \sup\{z \mid x \otimes z \leq y\}$. We will show that this supremum is in fact a maximum (that is, $x \otimes (x \Rightarrow y) \leq y$). Since $\otimes$ is continuous and non-decreasing, it commutes with suprema. It then follows that

$$x \otimes (x \Rightarrow y) = x \otimes \sup\{z \mid x \otimes z \leq y\} = \sup\{x \otimes z \mid x \otimes z \leq y\} \leq y.$$

Uniqueness follows trivially from the fact that every element is $\leq$ to itself. □

Exercise. Prove Lemma 2.2.

The operator $\Rightarrow$ defined in Lemma 2.2 is called the *residuum* of the t-norm $\otimes$. We now see that residua in fact satisfy some important properties that hold in classical logic.

Exercise. Show that for every continuous t-norm and its residuum $\Rightarrow$, and every $x, y \in [0,1]$

1. $x \leq y$ iff $(x \Rightarrow y) = 1$,
2. $(1 \Rightarrow x) = x$.

For the three main t-norms that we have defined, the residua are defined as follows.

Proposition 2.3. *The following operators define the residua of the three main t-norms: for $x > y$,*

1. *Łukasiewicz implication: $x \Rightarrow y = 1 - x + y$*
2. *Product implication: $x \Rightarrow y = y/x$ (also called Goguen implication)*
3. *Gödel implication: $x \Rightarrow y = y$.*

and $x \Rightarrow y = 1$ if $x \leq y$.

Proof. Let $1 \geq x > y$; then

1. $x \otimes z \leq y$ iff $x + z - 1 \leq y$ iff $z \leq 1 - x + y$; thus $1 - x + y = \max\{z \mid x \otimes z \leq y\}$.
2. $x \otimes z \leq y$ iff $x \cdot z \leq y$ iff $z \leq y/x$.
3. $x \otimes z \leq y$ iff $\min\{x, z\} \leq y$ iff $z \leq y$.

□

As mentioned before, the Gödel t-norm, defined as the minimum of its two arguments is the "natural" characterization of conjunction, and is in fact one of the most studied, both, because of its simplicity, and because it is very closely related to classical logic. This expressivity, however, can be simulated by any continuous t-norm.

Lemma 2.4. *For every continuous t-norm $\otimes$, and $x, y \in [0,1]$, the following hold:*

- $\min\{x, y\} = x \otimes (x \Rightarrow y)$
- $\max\{x, y\} = \min\{((x \Rightarrow y) \Rightarrow y), ((y \Rightarrow x) \Rightarrow x)\}$.

Proof. If $x \leq y$ then $(x \Rightarrow y) = 1$ and hence $x \otimes (x \Rightarrow y) = x$. If $x > y$ then by definition $(x \Rightarrow y) = \max\{z \mid x \otimes z \leq y\}$; thus $x \otimes (x \Rightarrow y) \leq y$. Assume that $x \otimes (x \Rightarrow y) < y$, then by continuity of $\otimes$, there exists a z' such that $x \otimes (x \Rightarrow y) < x \otimes z' \leq y$, which is a contradiction. Thus $x \otimes (x \Rightarrow y) = y$.

For the maximum, let $x \leq y$. Then $(x \Rightarrow y) \Rightarrow y = 1 \Rightarrow y = y$. Moreover, by definition of the residuum, we have that $y \otimes (y \Rightarrow x) \leq x$. This implies that $(y \Rightarrow x) \Rightarrow x = \max\{z \mid z \otimes (y \Rightarrow x) \leq x\} \geq y$, and so $\min\{((x \Rightarrow y) \Rightarrow y), ((y \Rightarrow x) \Rightarrow x)\} = y$. The case where $x \geq y$ is dual. □

One can additionally define the ordering $\leq$: $x \leq y$ iff $\min\{x, y\} = x$.

We have thus far defined an interpretation of the conjunction, and its associated residuum that will interpret the implication. With the help of the latter one, we will introduce a negation function. Notice that in classical logic one can define the negation $\neg\phi$ as $\phi \to false$. We use this same intuition, and interpret the negation using the *precomplement* of the residuum $\Rightarrow$.

Definition 2.5. Every residuum defines a corresponding unary operator $\ominus x = x \Rightarrow 0$, called the *precomplement.* △

Exercise. Find the precomplement of the three main continuous t-norms.

So far, only three different continuous t-norms have been introduced. We will now show how to construct new continuous t-norms by combining previously known t-norms into an ordinal sum.

Definition 2.6. Let $\mathcal{I}$ be a (possibly infinite) set of indices and (a_i, b_i), $i \in \mathcal{I}$ a family of pairwise disjoint, non-empty open subintervals of

$[0,1]$. The *ordinal sum* of the family $\otimes_i, i \in \mathcal{I}$ of t-norms is the function $\otimes : [0,1] \times [0,1] \to [0,1]$ defined by

$$x \otimes y = \begin{cases} a_i + (b_i - a_i) \cdot (\frac{x-a_i}{b_i-a_i} \otimes_i \frac{y-a_i}{b_i-a_i}) & \text{if } x, y \in [a_i, b_i] \\ \min\{x,y\} & \text{otherwise} \end{cases}$$

and denoted as $\sum_{i\in\mathcal{I}}(\otimes_i, a_i, b_i)$ △

The main intuition behind an ordinal sum is that each t-norm $\otimes_i$ is used locally over the interval $[a_i, b_i]$, and these are combined using the operator min. If all the t-norms $\otimes_i$ are continuous, then their ordinal sum is also a continuous t-norm.

Proposition 2.7. *If $\otimes_i$ is a continuous t-norm, for all $i \in \mathcal{I}$, then $\sum_{i\in\mathcal{I}}(\otimes_i, a_i, b_i)$ is also a continuous t-norm.*

Proof. Let $\otimes$ denote the ordinal sum $\sum_{i\in\mathcal{I}}(\otimes_i, a_i, b_i)$.

We first show that it is a t-norm. Commutativity and associativity are trivial consequences from the definition. So we only need to prove that it is non-decreasing and that 1 is a unit.

[Unit] Let $x \in [0,1]$. If there is some $i \in \mathcal{I}$ such that $x \in [a_i, b_i]$ and $b_i = 1$, then

$$\begin{aligned} 1 \otimes x &= a_i + (1-a_i) \cdot (\frac{1-a_i}{1-a_i} \otimes_i \frac{x-a_i}{1-a_i}) \\ &= a_i + (1-a_i) \cdot (1 \otimes_i \frac{x-a_i}{1-a_i}) \\ &= a_i + (1-a_i) \cdot (\frac{x-a_i}{1-a_i}) \\ &= a_i + x - a_i = x. \end{aligned}$$

Otherwise (that is, if there is no i with these properties) then

$$1 \otimes x = \min\{1, x\} = x.$$

[Non-decreasing] By commutativity, it suffices to show that if $x_1 \leq x_2$, then $x_1 \otimes y \leq x_2 \otimes y$. We will prove this with the help of the following claim.

Claim. *for every $x \in [0,1]$*

1. *if there is some $i \in \mathcal{I}$ such that $x \in [a_i, b_i]$, then for every $y \geq a_i, x \otimes y \in [a_i, b_i]$ and for every $y < a_i, x \otimes y = y$;*
2. *if there is no $i \in \mathcal{I}$ such that $x \in [a_i, b_i]$, then for every $y \in [0,1]$ $x \otimes y = \min\{x, y\}$.*

Proof of Claim. Point 2. and the second part of Point 1. are trivial from the definition of $\otimes$. For the remaining part of Point 1., if $y > b_i$, then $x \otimes y = x \in [a_i, b_i]$; otherwise,

$$\begin{aligned} 0 = \frac{a_i - a_i}{b_i - a_i} \otimes_i \frac{a_i - a_i}{b_i - a_i} &\leq \frac{x - a_i}{b_i - a_i} \otimes_i \frac{y - a_i}{b_i - a_i} \\ &\leq \frac{b_i - a_i}{b_i - a_i} \otimes_i \frac{b_i - a_i}{b_i - a_i} = 1 \end{aligned}$$

and thus $a_i = a_i + (b_i - a_i) \cdot 0 \leq x \otimes y \leq a_i + (b_i - a_i) \cdot 1 = b_i$. □

To show that $\otimes$ is non-decreasing, assume first that there is some $i \in \mathcal{I}$ such that $x_i, x_2 \in [a_i, b_i]$. Then, if $y \notin [a_i, b_i]$,

$$x_1 \otimes y = \min\{x_1, y\} \leq \min\{x_2, y\} = x_2 \otimes y.$$

If $y \in [a_i, b_i]$ then

$$\begin{aligned} x_1 \otimes y &= a_i + (b_i - a_i) \cdot (\frac{x_1 - a_i}{b_i - a_i} \otimes_i \frac{y - a_i}{b_i - a_i}) \\ &\leq a_i + (b_i - a_i) \cdot (\frac{x_2 - a_i}{b_i - a_i} \otimes_i \frac{y - a_i}{b_i - a_i}) \\ &= x_2 \otimes y. \end{aligned}$$

Assume now that there is no $i \in \mathcal{I}$ where $\{x_1, x_2\} \subseteq [a_i, b_i]$. We define the sets S_1, S_2 as follows:

$$S_k = \begin{cases} [a_j, b_j] & \text{if } x_k \in [a_j, b_j], j \in \mathcal{I} \\ x_k & \text{if for every } j \in \mathcal{I}, x_k \notin [a_j, b_j]. \end{cases}$$

Since the sets (a_i, b_i) are pairwise disjoint, and $x_1 \leq x_2$, it follows that $z_1 \leq z_2$ for every $z_1 \in S_1$ and $z_2 \in S_2$. Using the claim, it follows that $x_k \otimes y \in S_k \cup \{y\}$ and $x_k \otimes y \notin S_k$ iff for every $z \in S_k, y < z$. This all together implies that $x_1 \otimes y \leq x_2 \otimes y$.

We now prove that it is continuous. Due to commutativity and the continuity of the t-norms $\otimes_i$ it suffices to show that for every $y \in [0,1]$ and every $i \in \mathcal{I}$, the function $x \otimes y$ is continuous in a_i and b_i. Notice that if $a_i \leq y$, then $a_i \otimes y = a_i$ and if $y \leq a_i$ then $a_i \otimes y = y$. Likewise, if $y \leq b_i$ then $b_i \otimes y = y$ and if $b_i \leq y$ then $b_i \otimes y = b$. By the continuity of min we thus obtain continuity of $\otimes$. □

Conversely, it is clear that every continuous t-norm is the ordinal sum of continuous t-norms since every t-norm is a trivial ordinal sum of itself on the whole interval $(0,1)$. A more interesting result is given by the following theorem.

Theorem 2.8 (Mostert-Shields)**.** *Every continuous t-norm can be expressed as the ordinal sum of Łukasiewicz and product t-norms.*

This theorem justifies the study of only the three "main" t-norms that we have described before, as every other continuous t-norm is simply a combination of these three.

Interestingly, the residuum of a t-norm constructed as an ordinal sum can also be expressed through the residua of the t-norms.

Proposition 2.9. *Let $\otimes = \sum_{i\in\mathcal{I}}(\otimes_i, a_i, b_i)$, where each $\otimes_i$ is a continuous t-norm, and let $\Rightarrow_i$ denote the residuum of the t-norm $\otimes_i$. Then, the residuum of $\otimes$ is given by*

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ a_i + (b_i - a_i) \cdot (\frac{x-a_i}{b_i-a_i} \Rightarrow_i \frac{y-a_i}{b_i-a_i}) & \text{if } a_i < y < x \leq b_i, \\ y & \text{otherwise.} \end{cases}$$

Proof. Follows from a simple case analysis. □

2.2 The Basic Logic

We have said before that the t-norm will be used to interpret conjunctions; we have also claimed that the operator min is a natural generalization of the conjunction of classical logic. Since both are expressible together, rather than ignoring one of them, we will use them together, depending on the desired expressivity. Only the former is a primitive

constructor, while the latter, as shown in the previous section, is only an abbreviation of a longer formula.

Basically, given a continuous t-norm $\otimes$, we obtain a propositional calculus over the set of truth values $[0,1]$: $\otimes$ is the truth function for conjunction & and the residuum $\Rightarrow$ is the truth function of the implication $\rightarrow$. This is formally defined next.

Definition 2.10. Let $\otimes$ be a continuous t-norm. The propositional calculus $PC(\otimes)$ has a countable set of propositional variables $p_1, p_2, \ldots$, connectives & and $\rightarrow$ and the truth constant $\mathbf{0}$. The *formulas* of this calculus are defined in the usual way: every propositional variable is a formula, $\mathbf{0}$ is a formula, and if ϕ, ψ are formulas, then also $\phi \& \psi$ and $\phi \Rightarrow \psi$ are formulas.

A *valuation* is a mapping $\mathcal{V}$ assigning to each propositional variable p a value $\mathcal{V}(p) \in [0,1]$. This valuation is extended to arbitrary formulas as follows:

$$\begin{aligned} \mathcal{V}(\mathbf{0}) &= 0, \\ \mathcal{V}(\phi\&\psi) &= \mathcal{V}(\phi) \otimes \mathcal{V}(\psi), \\ \mathcal{V}(\phi \rightarrow \psi) &= \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\psi). \end{aligned}$$

△

Other interesting connectives that can be introduced as abbreviations of complex formulas are the following:

$$\begin{aligned} \phi \wedge \psi &:= \phi\&(\phi \rightarrow \psi), \\ \phi \vee \psi &:= ((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi) \\ \neg\phi &:= \phi \rightarrow \mathbf{0} \\ \phi \equiv \psi &:= (\phi \rightarrow \psi)\&(\psi \rightarrow \phi) \\ \mathbf{1} &:= \mathbf{0} \rightarrow \mathbf{0}. \end{aligned}$$

In particular, it follows that $\mathcal{V}(\phi\wedge\psi) = \min\{\mathcal{V}(\phi), \mathcal{V}(\psi)\}$ and $\mathcal{V}(\phi\vee\psi) = \max\{\mathcal{V}(\phi), \mathcal{V}(\psi)\}$ (see Lemma 2.4).

In some sense, we have two conjunction operators: the &-conjunction, which we will also call *strong conjunction*, that is interpreted with the t-norm, and the $\wedge$-conjunction (*weak conjunction*) interpreted with the operator min. On the other hand, we have only one disjunction, corresponding to the *weak disjunction* with max as its truth function. Although it is possible to also define a strong disjunction operator, this will extend the expressivity of the logic.

We will now develop a proof system for this logic, in which we are interested in deducing those formulas that are true under any evaluation. We call these 1-tautologies.

Definition 2.11. A formula ϕ is a *1-tautology* of $PC(\otimes)$ if $\mathcal{V}(\phi) = 1$ for every valuation $\mathcal{V}$. △

It should be clear that different t-norms $\otimes_1$ and $\otimes_2$ will produce a different set of 1-tautologies: $\neg\neg\phi \to \phi$ is a 1-tautology under the Łukasiewicz t-norm, but not under the product or Gödel t-norms [**Exercise?**]. We will for the moment abstract from these differences, and focus on finding formulas that are 1-tautologies for any continuous t-norm.

We proceed as follows: we will first present eight 1-tautologies that will be the basic axioms for all the logics $PC(\otimes)$. Together with modus ponens as a deduction rule, we obtain a sound logic. This means that every provable formula is itself a 1-tautology in every $PC(\otimes)$. In particular, this means that it is a tautology in classical logic. [Note, the converse is not true]

Definition 2.12. The following formulas are the axioms of the basic logic BL:

(A1) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

(A2) $(\phi \& \psi) \to \phi$

(A3) $(\phi \& \psi) \to (\psi \& \phi)$

(A4) $(\phi \& (\phi \to \psi)) \to (\psi \& (\psi \to \phi))$

(A5) $(\phi \to (\psi \to \chi)) \to ((\phi \& \psi) \to \chi)$

(A6) $((\phi \& \psi) \to \chi) \to (\phi \to (\psi \to \chi))$

(A7) $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$

(A8) $\mathbf{0} \to \phi$

The *deduction rule* of BL is modus ponens; that is, from ϕ and $\phi \to \psi$, we infer ψ.

A *proof* in BL is a sequence $\phi_1, \ldots, \phi_n$ of formulas such that every ϕ_i is either an axiom of BL or follows from some preceding ϕ_j, ϕ_k, $j, k < i$, by modus ponens. A formula ϕ is *provable* in BL, denoted as BL $\vdash \phi$ if there is a proof $\phi_1, \ldots, \phi_n$ such that $\phi_n = \phi$. △

Intuitively, the axioms of BL express the properties of t-norms and their residua: Axiom (A1) expresses the transitivity of implication; (A2) and (A3) express the monotonicity and commutativity of &-conjunction, respectively; (A4) expresses the commutativity of $\wedge$, (A5) and (A6) formulate the definition of residua and (A8) states that falsity implies everything. Finally, the axiom (A7) describes a variant of a proof by cases.

In order to have a sound proof system for the logic BL, it is necessary that all these axioms are in fact 1-tautologies in $PC(\otimes)$. Additionally, modus ponens should also be a sound deduction rule. This is given by the following lemma.

Lemma 2.13. *All the axioms of BL are 1-tautologies in $PC(\otimes)$ for every continuous t-norm $\otimes$. If ϕ and $\phi \to \psi$ are 1-tautologies of $PC(\otimes)$, then ψ is also a 1-tautology of $PC(\otimes)$.*

Proof. We will prove the lemma for the first four axioms only, the other four are left as an exercise.
[(A1)] To verify that this is a 1-tautology, we need to prove that for every $x, y, z \in [0, 1]$

$$1 \leq (x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)).$$

From the definition of residuum, this follows iff

$$(x \Rightarrow y) \leq (y \Rightarrow z) \Rightarrow (x \Rightarrow z).$$

Using the same argument, this is true iff $(y \Rightarrow z) \otimes (x \Rightarrow y) \leq x \Rightarrow z$ iff

$$(y \Rightarrow z) \otimes (x \Rightarrow y) \otimes x \leq z.$$

From Lemma 2.4, we then have

$$(y \Rightarrow z) \otimes (x \Rightarrow y) \otimes x \leq (y \Rightarrow z) \otimes y \leq z.$$

[(A2)] For every $x, y \in [0,1]$ it holds that $x \otimes y \leq x \otimes 1 = x$ and hence $(x \otimes y) \Rightarrow x = 1$.
[(A3)] From commutativity of $\otimes$, $x \otimes y = y \otimes x$ and hence $(x \otimes y) \Rightarrow (y \otimes x) = 1$.
[(A4)] From Lemma 2.4, $x \otimes (x \Rightarrow y) = \min\{x, y\} = \min\{y, x\} = y \otimes (y \Rightarrow x)$.

Finally, for soundness of modus ponens, recall that if $x = 1$, then $x \Rightarrow y = y$. Hence, if $x \Rightarrow y = 1$ it follows that $y = 1$. □

Exercise. Prove that (A5)-(A8) are 1-tautologies of $PC(\otimes)$.

A consequence of this lemma is that every formula that is provable in BL is also a 1-tautology in $PC(\otimes)$.

We now prove some properties of the different connectives. Notice that in the following we are interested more in the consequences that can be derived from the axioms through modus ponens than in proving that the formulas are 1-tautologies. Later on, we will prove a completeness theorem, that will show that provable in BL and 1-tautology in $PC(\otimes)$ are equivalent.

Lemma 2.14 (Implication)**.** *The following formulas are provable in BL:*

(1) $\phi \to (\psi \to \phi)$

(2) $(\phi \to (\psi \to \chi)) \to (\psi \to (\phi \to \chi))$

(3) $\phi \to \phi$

Proof. (1) From axioms (A2) and (A6), we have

$$(\phi \& \psi) \to \phi \text{ and } ((\phi \& \psi) \to \phi) \to (\phi \to (\psi \to \phi)).$$

Using modus ponens we obtain (1).
(2) $\mathsf{BL} \vdash ((\psi \& \phi) \to (\phi \& \psi)) \to (((\phi \& \psi) \to \chi) \to ((\psi \& \phi) \to \chi))$ [(A1)], thus $\mathsf{BL} \vdash ((\phi \& \psi) \to \chi) \to ((\psi \& \phi) \to \chi)$ [(A3) and modus ponens], and from (A5) and (A6) we get

$$\mathsf{BL} \vdash [\phi \to (\psi \to \chi)] \to [(\phi \& \psi) \to \chi] \to [(\psi \& \phi) \to \chi] \to [\psi \to (\phi \to \chi)]$$

From transitivity of implication, the rest follows. □

Exercise. Prove (3)

Lemma 2.15 (Strong conjunction)**.** *BL proves*

(4) $(\phi \& (\phi \to \psi)) \to \psi$

(5) $\phi \to (\psi \to (\phi \& \psi))$

(6) $(\phi \to \psi) \to ((\phi \& \chi) \to (\psi \& \chi))$

Proof. (4) $\mathsf{BL} \vdash (\phi \to \psi) \to (\phi \to \psi)$ [(3)] and so
$\mathsf{BL} \vdash \phi \to ((\phi \to \psi) \to \psi)$ [(2)]. Using (A5) we then get (4)
(5) From (3) $\mathsf{BL} \vdash (\phi \& \psi) \to (\phi \& \psi)$; using (A6) we have
$\mathsf{BL} \vdash \phi \to (\psi \to (\phi \& \psi))$.
(6) $\mathsf{BL} \vdash (\phi \& (\phi \to \psi)) \to \psi$ (4), $\mathsf{BL} \vdash \psi \to (\chi \to (\psi \& \chi))$ (5), so
$\mathsf{BL} \vdash (\phi \& (\phi \to \psi)) \to (\chi \to (\psi \& \chi))$ (A1)
$\mathsf{BL} \vdash \phi \to ((\phi \to \psi) \to (\chi \to (\psi \& \chi)))$ (A6)
$\mathsf{BL} \vdash \phi \to (\chi \to ((\phi \to \psi) \to (\psi \& \chi)))$ (2)
$\mathsf{BL} \vdash (\phi \& \chi) \to ((\phi \to \psi) \to (\psi \& \chi))$ (A5)
$\mathsf{BL} \vdash (\phi \to \psi) \to ((\phi \& \chi) \to (\psi \& \chi))$ (2) □

Exercise. Prove the associativity rules for strong conjunction and

$$((\phi_1 \to \psi_1) \& (\phi_2 \to \psi_2)) \to ((\phi_1 \& \phi_2) \to (\psi_1 \& \psi_2))$$

Lemma 2.16 (Weak conjunction)**.** *BL proves*

(7) $(\phi \& \psi) \to (\phi \wedge \psi)$

(8) $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$

Proof. (7) $\mathsf{BL} \vdash \psi \to (\phi \to \psi)$ (1)
$\mathsf{BL} \vdash (\psi \to (\phi \to \psi)) \to (\psi \& \phi \to (\phi \to \psi) \& \phi)$ (6). The rest is (A3) and (A1).
(8) $\mathsf{BL} \vdash (\psi \to \chi) \to (\psi \to (\psi \wedge \chi))$ (Exercise!) and
$\mathsf{BL} \vdash \psi \to (\psi \& (\psi \to \chi)) \to ((\phi \to \psi) \& ((\phi \to \psi) \to (\phi \to \chi)) \to (\phi \to (\psi \& (\psi \to \chi))))$; thus
$\mathsf{BL} \vdash (\psi \to (\psi \wedge \chi)) \to [((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))]$. From this, it follows that $\mathsf{BL} \vdash (\psi \to \chi) \to (8)$.
Analogously, $\mathsf{BL} \vdash (\chi \to \psi) \to (8)$. And from Axiom (A7), the result follows. □

Exercise. Complete the proof of Lemma 2.16 by showing that BL proves

$$(\phi \to \psi) \to (\phi \to (\phi \wedge \psi)).$$

Lemma 2.17 (Negation). *BL proves*

(9) $\phi \to (\neg\phi \to \psi)$

(10) $(\phi \to (\psi \& \neg\psi)) \to \neg\phi$

Proof. (9) $\mathsf{BL} \vdash \phi \to ((\phi \to \mathbf{0}) \to \mathbf{0})$ follows from (4) and $\mathsf{BL} \vdash \mathbf{0} \to \psi$ (A8) implies that $\mathsf{BL} \vdash ((\phi \to \mathbf{0}) \to \mathbf{0}) \to ((\phi \to \mathbf{0}) \to \psi)$. From this, we get (9).
(10) $\mathsf{BL} \vdash (\psi \& (\psi \to \mathbf{0})) \to \mathbf{0}$ (4). And so we get (10) □

Exercise. Prove the following in BL:

- $\phi \to \neg\neg\phi$,
- $(\phi \& \neg\phi) \to \mathbf{0}$. [This is used in the proof before, maybe too easy?]

In general, one is usually interested in finding the consequences of additional information (formulas), rather than the set of tautologies derivable from the calculus. This motivates the notion of a theory. Simply, a theory is a set of formulas that are assumed to be true; that is, considered "special axioms", and the deduction system is used to obtain the set of formulas that are necessary true in this theory. Obviously, this set will include all the tautologies of the original calculus.

Definition 2.18. A *theory* over BL is a set of formulas. A *proof* in a theory T is a sequence of formulas, where each element is either an axiom, an element of T, or follows from preceding elements through modus ponens. The formula ϕ is *provable* in T (denoted as $T \vdash \phi$) if it is the last element of a proof in T. △

As in classical logic, we can focus on studying only the tautologies of the logic, since the elements of the theory T can be transfered to the formula to be proven, as stated in the following theorem.

Theorem 2.19. *Let T be a theory and let ϕ, ψ be formulas. $T \cup \{\phi\} \vdash \psi$ iff there is an $n \in \mathbb{N}$ such that $T \vdash \phi^n \to \psi$ (where ϕ^n denotes $\phi \& ... \& \phi$ n times).*

Proof. [if] Since & is commutative and associative, for any $n > 1$ it follows that if $T \vdash \phi^n \to \psi$, then $T \vdash (\phi \& \phi^{n-1}) \to \psi$ and $T \vdash (\phi \to (\phi^{n-1} \to \psi)$, which by definition means that $T \cup \{\phi\} \vdash \phi^{n-1} \to \psi$. Repeating the same argument, we obtain $T \cup \{\phi\} \vdash \phi \to \psi$ and thus $T \cup \{\phi\} \vdash \psi$.

[only if] Assume now that $T \cup \{\phi\} \vdash \psi$ and let $\gamma_1, \ldots, \gamma_k$ be a proof for ψ. We prove by induction that, for each $j = 1, \ldots, k$, there is an n_j such that $T \vdash \phi^{n_j} \to \gamma_j$. If γ_j is an axiom of BL or a formula in $T \cup \{\phi\}$ this result trivially follows. If γ_j is obtained by applying modus ponens to previous formulas in the proof $\gamma_i, (\gamma_i \to \gamma_j)$, then by the induction hypothesis we can assume that $T \vdash \phi^n \to \gamma_i$ and $T \vdash \phi^m \to (\gamma_i \to \gamma_j)$ and hence [exercise given before] $T \vdash (\phi^n \& \phi^m) \to (\gamma_i \& (\gamma_i \to \gamma_j))$. Thus, $T \vdash \phi^{n+m} \to \gamma_j$ (4). □

Notice that in classical logic, the deduction theorem has a stronger form, in the sense that there is no need for conjoining the formula ϕ repeatedly with itself. This is mainly due to the idempotency of conjunction in classical logic, which does not hold in fuzzy logic in general. However, when the Gödel t-norm is used, then strong conjunction is again idempotent, and the stronger version of the deduction theorem holds.

Once we allow arbitrary formulas to be assumed to hold through a theory T, we encounter the risk of including contradictory information.

Definition 2.20. A theory T is *inconsistent* if $T \vdash \mathbf{0}$. Otherwise, it is *consistent.* △

Our logic BL cannot handle inconsistent theories, since every formula is provable from such a theory.

Lemma 2.21. *T is inconsistent iff $T \vdash \phi$ for all ϕ.*

Proof. If T proves every formula, then it proves $\mathbf{0}$. Conversely, if $T \vdash \mathbf{0}$, then from (A7) it follos that $T \vdash \phi$. □

Another characterization of inconsistency in classical logic is that if a formula makes a theory inconsistent then the theory could derive the negation of the formula. In general for fuzzy logics, only a weaker version of this result holds.

Lemma 2.22. *If $T \cup \{\phi\}$ is inconsistent, then $T \vdash \neg(\phi^n)$ holds for some n.*

Proof. If $T \cup \{\phi\} \vdash \mathbf{0}$, then by the deduction theorem, there is an n such that $T \vdash \phi^n \to \mathbf{0}$. $\square$

We have thus far developed a sound proof system, and used it to derive some interesting tautologies. It still remains to show that this system is complete; i.e., that every 1-tautology is provable in the system. To do this, we will first make a detour through residuated lattices, and remove the assumption that the truth values belong to a total order.

2.3 Residuated Lattices

We have defined, for every continuous t-norm $\otimes$ a propositional calculus $PC(\otimes)$ and described a sound logic BL that describes all the 1-tautologies in every $PC(\otimes)$.

We will now proceed with an algebraization of this logic. We will introduce a class of lattices called BL-algebras and show the following:

(i) for every t-norm $\otimes$, the interval $[0, 1]$ with the truth functions of connectives is a linearly ordered BL-algebra,

(ii) BL is sound for every linearly ordered BL-algebra; that is, every provable formula is a 1-tautology over such a lattice,

(iii) the set of all formulas (modulo provable equivalence) with the operations of the connectives is a BL-algebra, and

(iv) a tautology over all linearly ordered BL-algebras is a tautology over all BL-algebras.

To prove completeness of the logic w.r.t. any BL-algebra, we will then only need to prove the converse of the implication in (ii); the result then follows directly from (iv).

The lattices that define the BL-algebras will be a special kind of residuated lattices.

Definition 2.23. A *residuated lattice* is an algebra of the form

$$(L, \cap, \cup, \otimes, \Rightarrow, 0, 1)$$

such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with largest element 1 and least element 0,

(ii) $(L, \otimes, 1)$ is a commutative semigroup with unit 1, and

(iii) $\otimes$ and $\Rightarrow$ satisfy $z \leq (x \Rightarrow y)$ iff $x \otimes z \leq y$ for all $x, y, z \in L$.

The residuated lattice is *linearly ordered* if its lattice ordering is linear; that is, for every $x, y \in L$ $x \cap y = x$ or $x \cap y$. $\triangle$

For a residuated lattice to be a BL-algebra, we will require that the supremum and infimum lattice operators $(\cap, \cup)$, can be defined through the t-norm and its residuum, as done in Lemma 2.4.

Definition 2.24. A residuated lattice $(L, \cap, \cup, \otimes, \Rightarrow, 0, 1)$ is called a *BL-algebra* iff the following two identities hold for all $x, y \in L$.

1. $x \cap y = x \otimes (x \Rightarrow y)$,

2. $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

$\triangle$

This last restriction is called the axiom of *prelinearity*. Intuitively, if we have a total order, this axiom states that either $x \leq y$ or $y \leq x$. For general lattices the notion is somehow more complex, and will be explained later.

We now prove some basic properties of BL-algebras, that relate them to the properties of t-norms over the interval $[0, 1]$ studied before. Recall that in a lattice, the ordering $\leq$ is defined by $x \leq y$ iff $x \cap y = x$.

Lemma 2.25. *For each BL-algebra and $x, y, z \in L$, the following holds:*

1. *$x \otimes (x \Rightarrow y) \leq y$ and $x \leq (y \Rightarrow (x \otimes y))$,*

2. *$x \leq y$ implies $x \otimes z \leq y \otimes z$, $(z \Rightarrow x) \leq (z \Rightarrow y)$, $(y \Rightarrow z) \leq (x \Rightarrow z)$,*

3. $x \leq y$ *iff* $x \Rightarrow y = 1$,

4. $(x \cup y) \otimes z = (x \otimes z) \cup (y \otimes z)$,

5. $(x \cup y) = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$.

Proof. 1. Since $x \Rightarrow y \leq x \Rightarrow y$, it follows that $x \otimes (x \Rightarrow y) \leq y$. Additionally, since $x \otimes y \leq x \otimes y$, it follows that $x \leq y \Rightarrow (x \otimes y)$.

2. Let $x \leq y$. By 1. we have $y \leq z \Rightarrow (y \otimes z)$. Thus, $x \leq z \Rightarrow (y \otimes z)$, and $x \otimes z \leq y \otimes z$.
Additionally, we have $z \otimes (z \Rightarrow x) \leq x \leq y$ and so $(z \Rightarrow x) \leq (z \Rightarrow y)$
and also $x \otimes (y \Rightarrow z) \leq y \otimes (y \Rightarrow z) \leq z$ and thus $(y \Rightarrow z) \leq (x \Rightarrow z)$.

3. $x \leq y$ iff $1 \otimes x \leq y$ iff $1 \leq (x \Rightarrow y)$ iff $1 = (x \Rightarrow y)$.

4. Since $x \leq x \cup y$ we have that $x \otimes z \leq (x \cup y) \otimes z$. Likewise, $y \otimes z \leq (x \cup y) \otimes z$. Thus (lattice properties) $(x \otimes z) \cup (y \otimes z) \leq (x \cup y) \otimes z$. Conversely, $x \otimes z \leq (x \otimes z) \cup (y \otimes z)$ and thus $x \leq z \Rightarrow [(x \otimes z) \cup (y \otimes z)]$. Likewise, $y \leq z \Rightarrow [(x \otimes z) \cup (y \otimes z)]$, and hence $(x \cup y) \leq z \Rightarrow [(x \otimes z) \cup (y \otimes z)]$, and so $(x \cup y) \otimes z \leq (x \otimes z) \cup (y \otimes z)$.

5. From 4. it follows that
$(x \Rightarrow y) \otimes (x \cup y) = (x \otimes (x \Rightarrow y)) \cup (y \otimes (x \Rightarrow y)) \leq y \cup y = y$.
Then, $x \cup y \leq (x \Rightarrow y) \Rightarrow y$. Analogously, $x \cup y \leq (y \Rightarrow x) \Rightarrow x$, and so $x \cup y \leq ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$.
Conversely, since $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$, it follows that
$[((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)] = [\cdots] \otimes (x \Rightarrow y) \cup (y \Rightarrow x)$
$= ([\cdots] \otimes (x \Rightarrow y)) \cup ([\cdots] \otimes (y \Rightarrow x))$
$\leq [((x \Rightarrow y) \Rightarrow y) \otimes (x \Rightarrow y)] \cup [((y \Rightarrow x) \Rightarrow x) \otimes (y \Rightarrow x)]$
$\leq y \cup x$.

□

If we restrict to linearly ordered lattices (that is, where there are no incomparable elements), then the first condition from the definition of BL-algebras suffices.

Lemma 2.26. *A linearly ordered residuated lattice is a BL-algebra iff* $x \cap y = x \otimes (x \Rightarrow y)$ *holds for every* $x, y \in L$.

Proof. The "only if" direction is trivial by definition. For the converse, we only need to prove prelinearity. Since L is linearly ordered, it holds that $x \leq y$ or $y \leq x$ and hence $x \Rightarrow y = 1$ or $y \Rightarrow x = 1$ and hence $(x \Rightarrow y) = 1 \cup (y \Rightarrow x) = 1$. □

We say that a linearly ordered residuated lattice is *divisible* if for every x, y with $x > y$ there exists some z such that $y = x \otimes z$.

Exercise. Show that every divisible linearly ordered residuated lattice is a BL-algebra. [Hint: use Lemma 2.26]

It thus follows that every continuous t-norm on the interval $[0, 1]$ with the standard ordering of real numbers determines a BL-algebra. [Recall point (i) of the roadmap]

If we want to talk about tautologies in BL-algebras, we first need an appropriate notion of a valuation of the formulas. This is defined as for continuous t-norms over $[0, 1]$ (Definition 2.10).

Definition 2.27. Let $\mathbf{L} = (L, \cap, \cup, \otimes, \Rightarrow, 0, 1)$ be a BL-algebra. An $\mathbf{L}$-*valuation* is a mapping $\mathcal{V}$ assigning to each propositional variable p a value $\mathcal{V}(p) \in L$. This function is extended to arbitrary formulas in the usual way; that is,

$$\begin{aligned} \mathcal{V}(\mathbf{0}) &= 0, \\ \mathcal{V}(\phi \& \psi) &= \mathcal{V}(\phi) \otimes \mathcal{V}(\psi), \\ \mathcal{V}(\phi \to \psi) &= \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\psi). \end{aligned}$$

A formula ϕ is an $\mathbf{L}$-*tautology* if $\mathcal{V}(\phi) = 1$ for every $\mathbf{L}$-valuation $\mathcal{V}$. △

Notice that in particular, since $\mathbf{L}$ is a BL-algebra, we also obtain that, for every valuation $\mathcal{V}$

$$\begin{aligned} \mathcal{V}(\phi \wedge \psi) &= \mathcal{V}(\phi) \cap \mathcal{V}(\psi), \\ \mathcal{V}(\phi \vee \psi) &= \mathcal{V}(\phi) \cup \mathcal{V}(\psi), \\ \mathcal{V}(\neg \phi) &= \mathcal{V}(\phi) \Rightarrow 0. \end{aligned}$$

The BL-logic defined in the previous section is also sound when using any BL-algebra. The proof is very similar to the one of Lemma 2.13, and thus we omit it here.

Theorem 2.28. *If ϕ is provable in BL, then ϕ is an $\mathbf{L}$-tautology for every BL-algebra $\mathbf{L}$. More generally, if T is a theory over BL and T proves ϕ, then for every BL-algebra $\mathbf{L}$ and every $\mathbf{L}$-valuation $\mathcal{V}$ of the propositional variables that assigns the value 1 to all the axioms of T we have that $\mathcal{V}(\phi) = 1$.*

This shows the second point from our roadmap; in fact it shows a stronger result, since it is not limited to linearly ordered BL-algebras. We now turn our attention to the third point; that is, that the classes of provably equivalent formulas define a BL-algebra.

Definition 2.29. Let T be a fixed theory over BL. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash \phi \equiv \psi$ (that is, formulas that are T-provably equivalent to ϕ). Denote as L_T the set of all the classes $[\phi]_T$. We define the algebra $\mathbf{L}_T := (L_T, \cap, \cup, \otimes, \Rightarrow, 0, 1)$ where:

$$\begin{aligned} 0 &:= [\mathbf{0}]_T, \\ 1 &:= [\mathbf{1}]_T, \\ [\phi]_T \otimes [\psi]_T &:= [\phi \& \psi]_T, \\ [\phi]_T \Rightarrow [\psi]_T &:= [\phi \to \psi]_T, \\ [\phi]_T \cap [\psi]_T &:= [\phi \wedge \psi]_T, \\ [\phi]_T \cup [\psi]_T &:= [\phi \vee \psi]_T. \end{aligned}$$

$\triangle$

Lemma 2.30. *$\mathbf{L}_T$ is a BL-algebra*

Proof. We need first to prove that $(L_T, \cap, \cup, 0, 1)$ is a lattice bounded by 0 and 1. The properties of lattice can be derived from the axioms and properties proven before. For instance, let us show that $[\phi]_T \cap [\phi]_T = [\phi_T]$ (the idempotency of $\cap$). We need to show that $T \vdash \phi \equiv \phi \wedge \phi$. Recall that $\phi \wedge \psi := \phi \& (\phi \to \psi)$; thus we know (A2) that $\mathsf{BL} \vdash (\phi \wedge \phi) \to \phi$. We now show that $\mathsf{BL} \vdash \phi \to (\phi \wedge \phi)$:

From (3) we know that $\mathsf{BL} \vdash \phi \to \phi$. And from (5) it follows that $\mathsf{BL} \vdash (\phi \to \phi) \to (\phi \to (\phi \& (\phi \to \phi)))$. Using modus ponens, we get the desired conclusion. To prove that it is bounded, notice that $\mathsf{BL} \vdash \mathbf{0} \wedge \phi \equiv \mathbf{0}$ and that $\mathsf{BL} \vdash \mathbf{1} \vee \phi \equiv \mathbf{1}$. To show that $(L_T, \otimes, 1)$ is a commutative semigroup, we simply use axiom (A3) and the associativity of strong conjunction. That 1 is the unit is shown as follows:
$\mathsf{BL} \vdash \mathbf{1} \to (\phi \to (\mathbf{1} \& \phi))$ [(5)] and since $\mathsf{BL} \vdash \mathbf{0} \to \mathbf{0}$ [(3)] we get the desired equivalence.

We now need to show that this lattice is residuated, with operators $\otimes$ and $\Rightarrow$. For this, observe that the lattice ordering $\leq$ satisfies that

$$[\phi]_T \leq [\psi]_T \text{ iff } T \vdash \phi \to \psi:$$

if $T \vdash \phi \to \psi$, then $T \vdash \phi \equiv (\phi \wedge \psi)$, which means that

$$[\phi]_T = [\phi]_T \cap [\psi]_T$$

and thus $[\phi]_T \leq [\psi]_T$. Conversely, if $[\phi]_T \leq [\psi]_T$, then $T \vdash \phi \equiv (\phi \wedge \psi)$ and thus, $T \vdash \phi \to \psi$ (since $T \vdash \phi \wedge \psi \to \psi$).

Using this observation we now show the adjointness property:
$[\chi]_T \leq [\phi]_T \Rightarrow [\psi]_T$ iff $T \vdash \chi \to (\phi \to \psi)$ iff $T \vdash (\chi \& \phi) \to \psi$ (A5,A6) iff $[\chi \& \phi]_T \leq [\psi]_T$. All this shows that $\mathbf{L}_T$ is a residuated lattice.

We need only to show that the two conditions of Definition 2.24 (of BL-algebras) also hold. The first condition is a direct consequence of the definition of the weak conjunction $\wedge$. Hence, we need only prove the second condition; that is, $\mathsf{BL} \vdash (\phi \to \psi) \vee (\psi \to \phi) \equiv \mathbf{1}$, or equivalently $\mathsf{BL} \vdash (\phi \to \psi) \vee (\psi \to \phi)$. Notice that $\mathsf{BL} \vdash \phi \to ((\phi \to \psi) \to \psi)$ [(A6) and (4)] and $\mathsf{BL} \vdash \phi \to ((\psi \to \phi) \to \phi)$ [(1)], thus we have $\mathsf{BL} \vdash \phi \to (\phi \vee \psi)$. From this it follows that $\mathsf{BL} \vdash (\phi \vee \psi) \to (\phi \to \psi) \vee (\psi \to \phi)$. Analogously, we get that $\mathsf{BL} \vdash (\psi \vee \phi) \to (\phi \to \psi) \vee (\psi \to \phi)$. And thus, from axiom (A7) we get the prelinearity property. $\square$

To show completeness of our proof system, we will use the notion of filters. These will allow us to characterize homomorphisms from residuated lattices to linearly ordered lattices. Notice that this notion is given for all residuated lattices, even if they do not satisfy the extra requirements of BL-algebras.

Definition 2.31. Let $\mathbf{L} = (L, \cap, \cup, \otimes, \Rightarrow, 0, 1)$ be a residuated lattice. A *filter* on $\mathbf{L}$ is a non-empty set $F \subseteq L$ such that for every $x, y \in L$ the following conditions hold:

- if $\{x, y\} \subseteq F$, then $x \otimes y \in F$, and

- if $x \in F$ and $x \leq y$, then $y \in F$.

F is called a *prime filter* if for every $x, y \in L$ it holds that $x \Rightarrow y \in F$ or $y \Rightarrow x \in F$. △

Given a filter F on a BL-algebra $\mathbf{L}$, we define the relation $\sim_F$ on $\mathbf{L}$ as

$$x \sim_F y \qquad \text{iff} \qquad \{(x \Rightarrow y), (y \Rightarrow x)\} \subseteq F.$$

This relation in fact defines a congruence relation, and hence its quotient algebra is well-defined. This quotient is also a BL-algebra.

Lemma 2.32. *Let* $\mathbf{L}$ *be a BL-algebra and* F *a filter on* $\mathbf{L}$*. The following two properties hold:*

(i) $\sim_F$ *is a congruence relation, and the quotient alegebra* $\mathbf{L}/\sim_F$ *is a BL-algebra,*

(ii) $\mathbf{L}/\sim_F$ *is linearly ordered iff* F *is a prime filter.*

Proof. (i) We first show that $\sim_F$ is an equivalence relation. Obviously, since $F \neq \emptyset$ and $x \leq 1$ for all $x \in L$, it holds that $1 \in F$. Thus, $x \Rightarrow x = 1 \in F$ which means that $x \sim_F x$. Reflexivity is trivial from the definition. So we need only to prove that it is transitive. This follows from the fact that $((\phi \to \psi)\&(\psi \to \chi)) \to (\phi \to \chi)$ is an $\mathbf{L}$-tautology (axioms (A1) and (A6)) and thus, for every $x, y, z \in L$ it holds that $((x \Rightarrow y) \otimes (y \Rightarrow z)) \leq (x \Rightarrow z)$. This means that if $\{(x \Rightarrow y), (y \Rightarrow z)\} \subseteq F$, then also $x \Rightarrow z \in F$. Thus, $x \sim_F y, y \sim_F z$ implies $x \sim_F z$.

Let now $x \sim_F y$. Then we know that $x \otimes (x \Rightarrow y) \leq y$ and hence $z \otimes x \otimes (x \Rightarrow y) \leq z \otimes y$. But then it follows that $x \Rightarrow y \leq (z \otimes x) \Rightarrow (z \otimes y)$ and since $x \Rightarrow y \in F$, then also $(z \otimes x) \Rightarrow (z \otimes y) \in F$. This means that $z \otimes x \sim_F z \otimes y$.

To show that $x \Rightarrow z \sim_F y \Rightarrow z$, it is sufficient to prove that $(y \Rightarrow x) \leq (x \Rightarrow z) \Rightarrow (y \Rightarrow z)$, or equivalently, that $(y \Rightarrow x) \otimes (x \Rightarrow z) \leq (y \Rightarrow z)$. This follows from the Point 1. of Lemma 2.25:
$y \otimes (y \Rightarrow x) \otimes (x \Rightarrow z) \leq x \otimes (x \Rightarrow z) \leq z$, and hence we have $y \Rightarrow x) \leq (x \Rightarrow z) \Rightarrow (y \Rightarrow z)$, which means that $x \Rightarrow z \sim_F y \Rightarrow z$. The proof that $z \Rightarrow x \sim_F z \Rightarrow y$ is analogous.

(ii) First we show that if F is a prime filter, then $L/\sim_F$ is linearly ordered. Let $x, y \in L$; since F is a prime filter, we know that $x \Rightarrow y \in F$ or $y \Rightarrow x \in F$. Assume that the former is the case. We will then show that $[x]_F \leq [y]_F$; i.e., $[x]_F \cap [y]_F = [x]_F$. We know that $x \otimes (x \Rightarrow y) \leq x \otimes (x \Rightarrow y)$ and hence $x \Rightarrow y \leq x \Rightarrow x \otimes (x \Rightarrow y)$ and thus, as F is a filter we have that $x \Rightarrow x \otimes (x \Rightarrow y) \in F$. Additionally,

$$x \otimes (x \Rightarrow y) \otimes (x \Rightarrow y) \leq x$$

and hence $x \Rightarrow y \leq x \otimes (x \Rightarrow y) \Rightarrow x$ which implies that $x \otimes (x \Rightarrow y) \Rightarrow x \in F$. We thus have that $x \sim_F x \otimes (x \Rightarrow y) \Rightarrow x$ and so $[x]_F \cap [y]_F = [x \otimes (x \Rightarrow y)]_F = [x]_F$.

Conversely, assume that $L/\sim_F$ is linearly ordered and let $x, y \in L$. As $L/\sim_F$ is linearly ordered, we know that either $[x]_F \leq [y]_F$ or $[y]_F \leq [x]_F$. Assume w.l.o.g. that the former is the case. This means that $[x \otimes (x \Rightarrow y)]_F = [x]_F$ and hence $x \Rightarrow x \otimes (x \Rightarrow y) \in F$. But then, as $x \Rightarrow x \otimes (x \Rightarrow y) \leq x \Rightarrow y$ (Lemma 2.25 Point 1.) it follows that $x \Rightarrow y \in F$. □

The next lemma shows that we can use prime filters to remove any element that is not the supremum of a BL-algebra.

Lemma 2.33. *Let* $\mathbf{L}$ *be a countable BL-algebra and* $a \in L$ *such that* $a \neq 1$*. Then, there exists a prime filter* F *on* $\mathbf{L}$ *such that* $a \notin F$*.*

Proof. Obviously, $\{1\}$ is a filter not containing a, however it is not prime. We will show that if F is a filter not containing a and $x, y \in L$ are such that $\{x \Rightarrow y, y \Rightarrow x\} \cap F = \emptyset$, thene there is a filter $F' \supset F$ not containing a but containing either $x \Rightarrow y$ or $y \Rightarrow x$. For this we first prove the following claim.

Claim. *The least filter containing* F *and an element* $z \in L$ *is*

$$F' := \{u \mid \exists v \in F. \exists n \in \mathbb{N}. v \otimes z^n \leq u\}.$$

Exercise. Prove this claim (that is, show that F' is a filter, and that every filter containing F and z must be a superset of F'.

We know that F does contains neither $(x \Rightarrow y)$ nor $(y \Rightarrow x)$. Let now F_1, F_2 be the smallest filters containing F and $x \Rightarrow y$, $y \Rightarrow x$, respectively. We claim that $a \notin F_1 \cap F_2$. Assume on the contrary that $a \in F_1 \cap F_2$. Then, for some $v \in F$ and natural number n, we have that $v \otimes (x \Rightarrow y)^n \leq a$ and $v \otimes (y \Rightarrow x)^n \leq a$. Thus,

$$a \geq v \otimes (x \Rightarrow y)^n \cup v \otimes (y \Rightarrow x)^n = v \otimes ((x \Rightarrow y)^n \cup (y \Rightarrow x)^n) = v \otimes 1 = v$$

but since $v \in F$ and F is a filter, it then follows that $a \in F$, which is a contradiction. Thus, it follows that $a \notin F_1$ or $a \notin F_2$.

As $\mathbf{L}$ is countable, we can arrange all the pairs (x, y) from L^2 into a sequence $\{(x_n, y_n) \mid n \in \mathbb{N}\}$. Let $F_0 = \{1\}$ and given a filter F_n not containing a, we construct $F_{n+1} \supseteq F_n$ as follows: if the smallest filter containing F_n and $x_n \Rightarrow y_n$ does not contain a, then that is F_{n+1}; otherwise, F_{n+1} is the smallest filter containing F_n and $y_n \Rightarrow x_n$, which as shown before cannot contain a. Then, the union

$$\bigcup_{n \in \mathbb{N}} F_n$$

is a prime filter. □

Notice that the previous lemma also holds for uncountable BL-algebras, but the proof would require much more elaborate arguments that are beyond the scope of this course.

Finally, we show that arbitrary BL-algebras can be expressed using only linearly ordered ones, through a direct product of their carrier sets and operators.

Lemma 2.34. *Every BL-algebra is a subalgebra of the direct product of a system of linearly ordered BL-algebras.*

Proof. For a BL-algebra $\mathbf{L}$, let $\mathcal{U}$ be the class of all prime filters on $\mathbf{L}$, and for each $F \in \mathcal{U}$, let $\mathbf{L}_F$ denote $\mathbf{L}/\sim_F$. We define $\mathbf{L}^* := \prod_{F \in \mathcal{U}} \mathbf{L}_F$. From Lemma 2.32, it follows that $\mathbf{L}^*$ is the direct product of linearly ordered residuated lattices (because each $\mathbf{L}_F$ is linearly ordered).

For every $x \in \mathbf{L}$, let $i(x)$ be the element $([x]_F)_{F \in \mathcal{U}}$ of $\mathbf{L}^*$. It is easy to see that this embedding preserves the operations; thus we need only to show that it is injective. Let $x, y \in L$ with $x \neq y$. Then either $x \not\leq y$ or $y \not\leq x$. Assume the former; then it follows that $x \Rightarrow y \neq 1$. By Lemma 2.33, there exists a prime filter F not containing $x \Rightarrow y$ which means that $[x]_F \neq [y]_F$ and therefore $i(x) \neq i(y)$. □

With the help of this lemma we can show the next point in our roadmap; namely, that a formula that is a tautology in every linearly ordered BL-algebra is a tautology on every (arbitrary) BL-algebra.

Lemma 2.35. *If a formula ϕ is an $\mathbf{L}$-tautology for every linearly ordered BL-algebra $\mathbf{L}$, then it is an $\mathbf{L}$-tautology for every BL-algebra $\mathbf{L}$.*

Proof. Let $\mathbf{L}$ be a BL-algebra. From Lemma 2.34 we know that every element $x \in L$ can be expressed as the tuple $([x]_F)_{F \in \mathcal{U}}$, where each $\mathbf{L}/\sim_F$ is a linearly ordered BL-algebra.

Since ϕ is an $\mathbf{L}$-tautology, if we replace all the variables in ϕ by elements of L and the logical connectives by their corresponding interpretation function, we obtain an element $\widehat{\phi}$ of L. Moreover, since ϕ is a tautology in every $\mathbf{L}/\sim_F, F \in \mathcal{U}$, we have that $[\widehat{\phi}]_F = [1]_F$ and hence $i(\widehat{\phi}) = 1 = i(1)$. As i is an injective mapping, we then have that $\widehat{\phi} = 1$ and thus ϕ is a $\mathbf{L}$-tautology. □

Theorem 2.36 (Completeness)**.** *BL is complete. That is, for every formula ϕ, the following three statements are equivalent:*

(i) ϕ is provable in BL,

(ii) for every linearly ordered BL-algebra $\mathbf{L}$, ϕ is an $\mathbf{L}$-tautology,

(iii) for every BL-algebra $\mathbf{L}$, ϕ is an $\mathbf{L}$-tautology.

Proof. That (i) implies (ii) is a consequence of Theorem 2.28; (ii) implies (iii) is Lemma 2.35. Thus, we need only prove that (iii) implies (i).

Assume that ϕ is an $\mathbf{L}$-tautology for every BL-algebra $\mathbf{L}$. From Lemma 2.30 it follows that the algebra $\mathbf{L}_{\mathsf{BL}}$ of the classes of equivalent formulas of BL is a BL-algebra, and hence ϕ is an $\mathbf{L}_{\mathsf{BL}}$-tautology. Let now $\mathcal{V}$ be the valuation setting $\mathcal{V}(p_i) = [p_i]_{\mathsf{BL}}$ for every propositional variable. It then follows that $\mathcal{V}(\phi) = [\phi]_{\mathsf{BL}} = [\mathbf{1}]_{\mathsf{BL}}$, and thus $\mathsf{BL} \vdash \phi \equiv \mathbf{1}$, which is the same as $\mathsf{BL} \vdash \phi$; that is, ϕ is provable in BL. □

This finishes the proof of completeness of the logic BL. To summarize, we have shown that BL-provable and $\mathbf{L}$-tautology in every BL-algebra $\mathbf{L}$ are equivalent concepts. This, however, is not exactly what our initial goal was. Initially, we have defined the BL logic only for the BL-algebras defined by continuous t-norms over the interval $[0, 1]$; let us call these *t-algebras*. It would then be desirable to get a stronger theorem stating:

a formula ϕ is BL-provable iff ϕ is a $\mathbf{L}$-tautology for every t-algebra $\mathbf{L}$.

We have shown the "only if" direction of this statement (Lemma 2.13). The question of whether the other direction holds is still an open problem: it is unknown whether BL describes a complete axiomatization of the logics defined by continuous t-norms, or whether there exists a formula that is a tautology in each of these logics, but not for some other BL-algebra.

Before leaving the study of the basic logic, we briefly look at *complete* theories. A theory T is called *complete* if for every pair of formulas ϕ, ψ, it holds that $T \vdash (\phi \to \psi)$ or $T \vdash (\psi \to \phi)$.

Lemma 2.37. *For every theory T, if $T \nvdash \phi$, then there exists a consistent complete supertheory $T' \supseteq T$ such that $T' \nvdash \phi$.*

The proof of this lemma is analogous to the one presented for prime filters (Lemma 2.33). [**Exercise?**]

We now take a step away from generalities, and focus on specific instances of the logic, defined by t-norms, and their properties.

3 T-norm Based Logics

We will now focus on the properties inherent to the fuzzy logic produced by specific t-norms: first the Łukasiewicz and then the product t-norm.

3.1 Łukasiewicz Logic

In this section we look at the specific propositional calculus $PC(\otimes_{\text{Ł}})$ defined by the Łukasiewicz t-norm $x \otimes y = \max(0, x+y-1)$. This t-norm has a precomplement function that is involutive; that is, $\ominus \ominus x = x$ for all $x \in [0, 1]$. This property will be useful for deriving some interesting consequences of the calculus.

As explained before, the formula $\neg\neg\phi \to \phi$ is in fact a 1-tautology in this calculus, although it is not a tautology in every BL-algebra (or, for that matter, in the algebra defined by other t-norms). In fact, it turns out that this axiom is all that is necessary to have a complete axiomatization of this calculus.

Definition 3.1. The *Łukasiewicz propositional logic* (denoted as Ł) is the theory that extends BL with the axiom

$(\neg\neg)$ $\neg\neg\phi \to \phi$.

$\triangle$

The soundness of this system is an easy consequence of the soundness of BL and the properties of the Łukasiewicz t-norm. Just as for the general BL logic, it is possible to prove completeness of this system in a general setting, based on a sub-class of BL-algebras called MV-algebras, which stands for "multi-valued" algebras.

Definition 3.2. An *MV-algebra* is a BL-algebra that additionally satisfies the identity $x = ((x \Rightarrow 0) \Rightarrow 0)$ for every $x \in L$.

This restriction is the obvious translation from the new axiom to the algebraic structure.

As done in the previous chapter, one can prove the following proposition.

Proposition 3.3 (Completeness). *A formula ϕ is provable in the logic Ł iff it is a 1-tautology of Ł.*

(Classical) propositional logic is *compact* in the sense that a formula is true in an infinite theory T then it is also true in a finite subtheory of T. This is not the case in the logic Ł.

Exercise. Let $T = \{np \to q \mid n \in \mathbb{N}\} \cup \{\neg p \to q\}$, where $1p = p$ and $(n+1)p = \neg(\neg p \& \neg np)$. Show that q is true in each model of T, but not in any finite submodel of T.

We have thus far focused on formulas that are absolutely true given a background theory. However, when reasoning about vagueness, a natural question that arises is whether one can also reason about partially true formulas: the theory may contain some partially true formulas, and one may be interested in proving that a conclusion is also partially true. For Łukasiewicz logic, this is possible, as we will show next.

The main observation is that, given a valuation $\mathcal{V}$, if $\mathcal{V}(\phi) = r$, then for every formula ψ it holds that $\mathcal{V}(\psi) \geq r$ iff $\mathcal{V}(\phi \to \psi) = 1$. Thus, we can try to introduce, for every rational number $r \in [0,1]$ the *truth constant* $\mathbf{r}$: a formula such that $\mathcal{V}(\mathbf{r}) = r$ for every valuation $\mathcal{V}$. We will thus have that $\mathcal{V}(\psi) \geq r$ iff $\mathcal{V}(\mathbf{r} \to \psi) = 1$ and $\mathcal{V}(\psi) \leq r$ iff $\mathcal{V}(\psi \to \mathbf{r}) = 1$.

Notice that the idea of having truth constants has been already used before: we have defined the constants $\mathbf{0}$ and $\mathbf{1}$ before. We are now only extending this to all possible rational numbers. Indeed, it would be possible to allow any number in $[0,1]$ as a constant, but that would produce an uncountable number of formulas, which is usually undesirable in a logic.

Definition 3.4. Rational Pavelka Logic (RPL) extends the logic Ł by adding the truth constant $\mathbf{r}$ for every rational number r. Formulas are built from propositional variables and truth constants using the same connectives as Ł (i.e., $\to, \neg, \&$, etc.).

A valuation of the propositional variables extends to arbitrary formulas in the obvious way, with $\mathcal{V}(\mathbf{r}) = r$ for every $\mathcal{V}, r \in \mathbb{Q}$.

The *axioms* of RPL are the axioms of Ł plus the following axioms for truth constants:

$$\begin{aligned} (\mathbf{r} \to \mathbf{s}) &\equiv \widehat{\mathbf{r} \Rightarrow \mathbf{s}} \\ \neg \mathbf{r} &\equiv \widehat{\mathbf{1} - \mathbf{r}}. \end{aligned}$$

The notions of theory, proof, and models are as usual.

A *graded formula* is a pair (ϕ, r), where ϕ is a formula and $r \in \mathbb{Q} \cap [0,1]$; it is just an abbreviation for the formula $(\mathbf{r} \to \phi)$; that is, (ϕ, r) is true iff ϕ is evaluated to a value greater or equal to r. △

Lemma 3.5. *If a theory T proves (ϕ, r) and $(\phi \to \psi, s)$, then it proves $(\psi, r \otimes s)$, where $\otimes$ is the Łukasiewicz t-norm.*

Proof. If $T \vdash \mathbf{r} \to \phi$ and $T \vdash \mathbf{s} \to (\phi \to \psi)$, then
$T \vdash (\mathbf{r} \& \mathbf{s}) \to (\phi \& (\phi \to \psi))$ and thus $T \vdash \widehat{r \otimes s} \to \psi$. □

Notice that the presence of an axiom in a theory T may fail to guarantee that a formula is true, but one may still deduce that its truth value has to be greater than some number r; in other words, assuming that all the formulas in T are true may not guarantee truth of ϕ but only of $(\mathbf{r} \to \phi)$. The same idea can also be applied to provability, as in the following definition.

Definition 3.6. Let T be a theory over RPL and ϕ a formula. The *truth degree* of ϕ over T is given by

$$\|\phi\|_T := \inf\{\mathcal{V}(\phi) \mid \mathcal{V} \text{ is a model of } T\}.$$

The *provability degree* of ϕ over T is given by $|\phi|_T := \sup\{r \mid T \vdash (\phi, r)\}$.

We will now proceed to prove completeness of RPL by showing that the provability and truth degrees of a formula over a theory coincide. Recall first the deduction theorem: $T \cup \{\phi\} \vdash \psi$ iff $T \vdash \phi^n \to \psi$ for some $n \in \mathbb{N}$, and that every consistent theory has a complete consistent supertheory (Lemma 2.37). These two results remain valid in RPL, since the presence of truth constants does not affect their proofs. We can now prove the following two lemmata.

Lemma 3.7. *If a theory T does not prove $(\mathbf{r} \to \phi)$, then $T \cup \{\phi \to \mathbf{r}\}$ is consistent.*

Proof. Assume on the contrary that $T \cup \{\phi \to \mathbf{r}\}$ is inconsistent, that is, $T \cup \{\phi \to \mathbf{r}\} \vdash \mathbf{0}$. By the deduction theorem, there is an n such that $T \vdash (\phi \to \mathbf{r})^n \to \mathbf{0}$. Recall that $T \vdash (\phi \to \mathbf{r})^n \vee (\mathbf{r} \to \phi)^n$ and hence $T \vdash \mathbf{0}^n \vee (\mathbf{r} \to \phi)^n$, which means that $T \vdash (\mathbf{r} \to \phi)^n$, yielding a contradiction. □

Lemma 3.8. *Let T be a consistent and complete theory. Then the following hold:*

(1) For every formula ϕ, $|\phi|_T = \sup\{r \mid T \vdash \mathbf{r} \to \phi\} = \inf\{s \mid T \vdash \phi \to \mathbf{s}\}$,

(2) The provability degree commutes with the connectives; that is,

$$|\neg\phi|_T = 1 - |\phi|_T, \qquad |\phi \to \psi|_T = |\phi|_T \Rightarrow |\psi|_T.$$

In particular, this means that the valuation $\mathcal{V}(p_i) = |p_i|_T$ is a model of T.

Proof. (1) Since for every r, either $T \vdash \phi \to \mathbf{r}$ or $T \vdash \mathbf{r} \to \phi$, it suffices to show that $T \vdash \mathbf{r} \to \phi$ and $T \vdash \phi \to \mathbf{s}$ implies $r \leq s$. Assume on the contrary that $r > s$. Then, we know that $r \Rightarrow s < 1$, but since T proves $\mathbf{r} \to \phi$ and $\phi \to \mathbf{s}$, it also proves $\mathbf{r} \to \mathbf{s}$; that is, $T \vdash \widehat{\mathbf{r} \Rightarrow \mathbf{s}}$. However, $r \Rightarrow s < 1$ implies that there is some n such that $(r \Rightarrow s)^n = 0$ and thus T is inconsistent.

(2) For $\neg$:

$$\begin{aligned} |\neg\phi| &= \sup\{r \mid T \vdash \mathbf{r} \to \neg\phi\} = \sup\{r \mid T \vdash \phi \to 1 - \mathbf{r}\} \\ &= \sup\{1 - s \mid T \vdash \phi \to \mathbf{s}\} = 1 - \inf\{s \mid T \vdash \phi \to \mathbf{s}\} \\ &= 1 - |\phi|. \end{aligned}$$

For $\Rightarrow$, we use the continuity and other properties of $\Rightarrow$:

$$\begin{aligned} |\phi| \Rightarrow |\psi| &= \inf\{r \mid T \vdash \phi \to \mathbf{r}\} \Rightarrow \sup\{s \mid T \vdash \mathbf{s} \to \psi\} \\ &= \sup\{r \Rightarrow \sup\{s \mid T \vdash \mathbf{s} \to \psi\} \mid T \vdash \phi \to \mathbf{r}\} \\ &= \sup\{r \Rightarrow s \mid T \vdash \phi \to \mathbf{r}, T \vdash \mathbf{s} \to \psi\} \\ &\leq \sup\{r \mid T \vdash \mathbf{r} \to (\phi \to \psi)\} = |\phi \to \psi| \end{aligned}$$

since $T \vdash \widehat{r \Rightarrow s} \to (\mathbf{r} \to \mathbf{s})$, and hence, if $T \vdash \phi \to \mathbf{r}$ and $T \vdash \mathbf{s} \to \psi$, then $T \vdash \widehat{r \Rightarrow s} \to (\phi \to \psi)$.

Suppose now that the inequality is strict; that is, $(|\phi| \Rightarrow |\psi|) < t < t' < |\phi \to \psi|$ for some rational numbers t, t'. We can then express t as $r \Rightarrow s$ for some $r < |\phi|, s > |\psi|$; then $T \vdash \mathbf{r} \to \phi, T \vdash \psi \to \mathbf{s}$, and hence $T \vdash (\phi \to \psi) \to (\mathbf{r} \to \mathbf{s})$, and $T \vdash (\phi \to \psi) \to \mathbf{t}$. Since $T \vdash \mathbf{t}' \to (\phi \to \psi)$, we have $T \vdash \mathbf{t}' \to \mathbf{t}$, or equivalently $T \vdash \widehat{t' \Rightarrow t}$. But $t' \Rightarrow t < 1$. This means that T is inconsistent. We thus have that the equality holds. □

The completeness of RPL is a simple consequence of these results.

Theorem 3.9. *For every theory T and formula ϕ, it holds that*

$$|\phi|_T = \|\phi\|_T.$$

Proof. The fact that $|\phi|_T \leq \|\phi\|_T$ can be shown using the same arguments for showing soundness of the proof system; we thus focus on showing only the other inequality.

To show that $\|\phi\|_T \leq |\phi|_T$, it suffices to prove that $T \vdash \mathbf{r} \to \phi$ for every $r < \|\phi\|_T$: if this holds, then

$$\begin{aligned} |\phi|_T &= \sup\{r \mid T \vdash \mathbf{r} \to \phi\} \\ &\geq \sup\{r \mid r < \|\phi\|_T\} \\ &\geq \|\phi\|_T. \end{aligned}$$

If $T \nvdash \mathbf{r} \to \phi$, then by Lemma 3.7 $T \cup \{\phi \to \mathbf{r}\}$ is consistent, and thus by our previous remark, has a consistent complete supertheory T'. From Lemma 3.8 it follows that the valuation $\mathcal{V}(p_i) = |p_i|_{T'}$ is a model of T' and $\mathcal{V}(\phi \to \mathbf{r}) = 1$; that is, $r \geq \mathcal{V}(\phi) \geq \|\phi\|_{T'} \geq \|\phi\|_T$. □

We have shown earlier that Ł (and hence RPL) is not compact, according to one classical formulation of compactness; however, we will see that a different formulation of compactness holds for this logic.

Theorem 3.10. *Let T be a theory over RPL. If every finite $T_0 \subseteq T$ has a model, then T has a model.*

Proof. Assume that T has not model. Then, since RPL is complete, T must be inconsistent, and thus $T \vdash \mathbf{0}$. In other words, there is a proof for $\mathbf{0}$ from this theory. Since proofs are finite, this means that only finitely many axioms from T were used; thus, there is a finite subtheory $T_0 \subseteq T$ that is inconsistent, and hence has no models. □

3.2 Product Logic

We now focus on the propositional calculus $PC(\otimes_\Pi)$, defined through the product t-norm $x \otimes y = x \cdot y$; we will call this *product logic* and denote it as Π. Recall that the corresponding residuum is the Goguen implication and the precomplement is Gödel negation. As before, we will extend the axiomatization of BL to obtain a complete proof system for this logic in relation to a special class of algebras.

Definition 3.11. The axioms of the logic Π are those of BL plus the two axioms

(Π1) $\neg\neg\chi \to ((\phi\&\chi \to \psi\&\chi) \to (\phi \to \psi))$,

(Π2) $\phi \wedge \neg\phi \to \mathbf{0}$.

Lemma 3.12. *The axioms* (Π1) *and* (Π2) *are 1-tautologies over the product t-norm.*

Proof. (Π1) Let $\mathcal{V}$ be a valuation. If $\mathcal{V}(\chi) = 0$, then $\mathcal{V}(\neg\neg\chi) = 0$ and $\mathcal{V}(\neg\neg\chi \to \zeta) = 1$. If $\mathcal{V}(\chi) > 0$, then $\mathcal{V}(\neg\neg\chi) = 1$. There are two cases: (i) if $\mathcal{V}(\phi\&\chi) \leq \mathcal{V}(\psi\&\chi)$, then $\mathcal{V}(\phi) \cdot \mathcal{V}(\chi) \leq \mathcal{V}(\psi) \cdot \mathcal{V}(\chi)$ and as $\mathcal{V}(\chi) > 0$ it follows that $\mathcal{V}(\phi) \leq \mathcal{V}(\psi)$. Thus, $\mathcal{V}(\phi\&\chi \to \psi\&\chi) = \mathcal{V}(\phi \to \psi) = 1$; (ii) if $\mathcal{V}(\phi\&\chi) > \mathcal{V}(\phi\&\chi)$, then $\mathcal{V}(\phi) > \mathcal{V}(\psi)$ and thus $\mathcal{V}(\phi\&\chi \to \psi\&\chi) = \mathcal{V}(\phi \to \psi) = \mathcal{V}(\psi)/\mathcal{V}(\phi)$.

(Π2) Since in Π $\neg$ is interpreted as the Gödel negation, either $\mathcal{V}(\phi) = 0$ or $\mathcal{V}(\neg\phi) = 0$.

□

Exercise. Show that the axiom (Π2) can be equivalently replaced by any of the following formulas:

$$\neg(\phi\&\phi) \to \neg\phi, \quad (\phi \to \neg\phi) \to \neg\phi, \quad \neg\phi \vee \neg\neg\phi.$$

We now restrict the class of BL-algebras to also satisfy the restrictions given by the two new axioms of Π.

Definition 3.13. A Π-*algebra* (or product algebra) is a BL-algebra that additionally satisfies:

- $\ominus \ominus z \leq ((x \otimes z \Rightarrow y \otimes z) \Rightarrow (x \Rightarrow y))$, and
- $x \cap \ominus x = 0$.

Notice that Π is trivially sound w.r.t. product algebras, since the new restrictions are simple translations of the axioms added to BL into the algebraic setting. We now show that these algebras also satisfy some properties we should expect from any generalization of the product t-norm.

Exercise. Prove that the following sentences hold in every linearly ordered product algebra:

(1) if $x > 0$ then $\ominus x = 0$,

(2) if $z > 0$ then $x \otimes z = y \otimes z$ implies $x = y$, and

(3) if $z > 0$ then $x \otimes z < y \otimes z$ implies $x < y$.

Just as for the logics BL and Ł, we can show a completeness theorem for the logic Π w.r.t. product algebras.

Proposition 3.14 (Completeness)**.** *(1) A formula ϕ is provable in the product logic Π iff it is a 1-tautology of Π.*

(2) Let T be a finite theory over Π and ϕ a formula. T proves ϕ over the logic Π iff ϕ is true in every model of T.

Łukasiewicz logic can be "embeded" into Product logic in the sense that the Łukasiewicz t-norm can be isomorphically transformed into a (restricted) product on the interval $[a, 1]$ for any arbitrary, but fixed, $a, 0 < a < 1$.

Lemma 3.15. *For every $a, 0 < a < 1$, the interval $[0, 1]$ with $\otimes_L$ is isomorphic to $[a, 1]$ with the restricted product t-norm $\otimes_a$ given by $x \otimes_a y = \max(a, x \cdot y)$.*

Proof. Given $a \in (0,1)$, we construct the isomorphism $f_a(x) = a^{1-x}$, which means that $f_a^{-1}(y) = 1 - \log_a y$. We have that

$$f_a(x) \otimes_a f_a(y) = \max(a, f_a(x) \cdot f_a(y)) = \max(a, a^{2-x-y}),$$

and $f_a(x \otimes_{\text{Ł}} y) = a^{1-\max(0,x+y-1)}$. In both cases, the result is a if $x + y \leq 1$ and $a^{1-(x+y-1)}$ otherwise. □

Let now p_0 be a *new* propositional variable. For each formula ϕ where p_0 does not appear, we define a translation ϕ^Π as follows:

$$\begin{aligned} \mathbf{0}^\Pi &:= p_0 \\ p_i^\Pi &:= p_0 \vee p_i \quad \text{for } i \neq 0 \\ (\phi\&\psi)^\Pi &:= p_0 \vee (\phi^\Pi \& \psi^\Pi) \\ (\phi \to \psi)^\Pi &:= \phi^\Pi \to \psi^\Pi. \end{aligned}$$

In particular, it holds that $(\neg\phi)^\Pi := \phi^\Pi \to p_0$. We show now a correspondence between valuations over the Łukasiewicz and product semantics by means of this translation.

Lemma 3.16. *Let $\mathcal{V}$ be a valuation of propositional variables (including p_0) where $\mathcal{V}(p_0) > 0$. Let $\mathcal{V}', \mathcal{V}''$ be the valuations given by $\mathcal{V}'(p_i) = \max(a, \mathcal{V}(p_i))$ and $\mathcal{V}''(p_i) = f_a^{-1}(\mathcal{V}'(p_i))$ with $a = \mathcal{V}(p_0)$. For every formula ϕ not containing p_0 it holds that $f_a(\mathcal{V}''_{Ł}(\phi)) = \mathcal{V}_\Pi(\phi^\Pi)$.*

Proof. We prove this by induction on the structure of ϕ.

For the atomic formulas, if $\phi = \mathbf{0}$, then $f_a(\mathcal{V}''_{\text{Ł}}(\mathbf{0})) = f_a(0) = a = \mathcal{V}(p_0) = \mathcal{V}_\Pi(\mathbf{0}^\Pi)$.
If $\phi = p$, then on the one hand, $\mathcal{V}_\Pi(p^\Pi) = \mathcal{V}(p_0 \vee p) = \max(a, \mathcal{V}(p))$ and on the other $f_a(\mathcal{V}''_{\text{Ł}}(p)) = f_a(f_a^{-1}(\max(a, \mathcal{V}(p))) = \max(a, \mathcal{V}(p))$.

Now for the complex formulas. Let first $\phi = \psi\&\chi$. Then

$$\begin{aligned} \mathcal{V}_\Pi((\psi\&\chi)^\Pi) &= \mathcal{V}_\Pi(p_0 \vee (\psi^\Pi \& \chi^\Pi)) = \max(a, \mathcal{V}_\Pi(\psi^\Pi \& \chi^\Pi)) \\ &= \max(a, \mathcal{V}_\Pi(\psi^\Pi) \cdot \mathcal{V}_\Pi(\chi^\Pi)) \end{aligned}$$

and $f_a(\mathcal{V}''_{\text{Ł}}(\psi\&\chi)) = f_a(\max(0, \mathcal{V}''_{\text{Ł}}(\psi) + \mathcal{V}''_{\text{Ł}}(\chi) - 1))$. By induction, this last expression is equal to

$$\begin{aligned} & f_a(\max(0, f_a^{-1}(\mathcal{V}_\Pi(\psi^\Pi)) + f_a^{-1}(\mathcal{V}_\Pi(\chi^\Pi)) - 1)) \\ &= f_a(\max(0, 1 - \log_a(\mathcal{V}_\Pi(\psi^\Pi) \cdot \mathcal{V}_\Pi(\chi^\Pi)))) \\ &= \max(a, \mathcal{V}_\Pi(\psi^\Pi) \cdot \mathcal{V}_\Pi(\chi^\Pi)). \end{aligned}$$

Finally, for $\phi = \psi \to \chi$ we have

$$\mathcal{V}_\Pi((\psi \to \chi)^\Pi) = \mathcal{V}_\Pi(\psi^\Pi \to \chi^\Pi),$$

which is 1 if $\mathcal{V}_\Pi(\psi^\Pi) \leq \mathcal{V}_\Pi(\chi^\Pi)$ and $\mathcal{V}_\Pi(\chi^\Pi)/\mathcal{V}_\Pi(\psi^\Pi)$ otherwise. On the other hand,

$$f_a(\mathcal{V}''_{\text{Ł}}(\psi \to \chi)) = f_a(\min(1, 1 - \mathcal{V}''_{\text{Ł}}(\psi) + \mathcal{V}''_{\text{Ł}}(\chi))),$$

which is 1 if $\mathcal{V}''_{\text{Ł}}(\psi) \leq \mathcal{V}''_{\text{Ł}}(\psi)$ (which holds iff $\mathcal{V}_\Pi(\psi^\Pi) \leq \mathcal{V}_\Pi(\chi^\Pi)$, from induction hypothesis plus monotonicity of f_a) and otherwise it is

$$\begin{aligned} & f_a(1 - f_a^{-1}(\mathcal{V}_\Pi(\psi^\Pi)) + f_a^{-1}(\mathcal{V}_\Pi(\chi^\Pi))) \\ &= f_a(1 + \log_a \mathcal{V}_\Pi(\psi^\Pi) - \log_a \mathcal{V}_\Pi(\chi^\Pi)) \\ &= \mathcal{V}_\Pi(\chi^\Pi)/\mathcal{V}_\Pi(\psi^\Pi). \end{aligned}$$

□

From these translations we obtain that tautologies in the logic Ł can be translated into tautologies in Π; that is, Ł can be embedded into Π.

Theorem 3.17. *Let ϕ be a formula not containing p_0. ϕ is a 1-tautology of Ł if and only if $(\neg\neg p_0) \to \phi^\Pi$ is a 1-tautology of* Π.

Proof. Suppose first that ϕ is a 1-tautology for Ł, and let $\mathcal{V}$ be a valuation. If $\mathcal{V}(p_0) = 0$, then $\mathcal{V}_\Pi(\neg\neg p_0 \to \phi^\Pi) = 1$. Otherwise, as ϕ is a tautology for Ł, we have that $\mathcal{V}''_{\text{Ł}}(\phi) = 1$ and thus, from Lemma 3.15 it follows that $\mathcal{V}_\Pi(\phi^\Pi) = 1$.

Conversely, if $\neg\neg p_0 \to \phi^\Pi$ is a 1-tautology for Π, then for every valuation $\mathcal{V}$ it holds that $\mathcal{V}_\Pi(\neg\neg p_0 \to \phi^\Pi) = 1$; in particular, for every valuation where $\mathcal{V}(p_0) > 0$, it holds that $\mathcal{V}_\Pi(\phi^\Pi) = 1$.

Claim. *For every valuation $\mathcal{V}$ there exists a valuation $\mathcal{W}$ and a constant $a, 0 < a < 1$ such that $\mathcal{V}(p) = f_a^{-1}(\max(a, \mathcal{W}(p)))$ for every propositional variable $p \neq p_0$.*

Exercise. Prove this claim.

Let now $\mathcal{W}(p_0) = a$. Then we have that $\mathcal{W}_\Pi(\phi^\Pi) = 1$ and thus, by Lemma 3.15, $\mathcal{V}_Ł(\phi) = 1$, which means that ϕ is a 1-tautology for Ł. □

Exercise. Show that every linearly ordered MV-algebra is an algebra of the form $MV'(\mathbf{L}, a)$ where $\mathbf{L}$ is a linearly ordered product algebra, $a \in \mathbf{L}$ with $0_\mathbf{L} < a < 1_\mathbf{L}$, the domain of $MV'(\mathbf{L}, a)$ is the interval $[a, 1_\mathbf{L}]$.

Intuitively, this theorem expresses that the product logic Π is more difficult than the Łukasiewicz logic Ł: everything that can be done in Π can also be done in Ł through via this embedding. However, the other direction does not hold. For instance, with Łukasiewicz logic, we were able to express also formulas that are partially true (RPL), but an analogous of RPL over Π is not possible.

Consider the system analogous to RPL over the logic Π and let $T = \{p \to \mathbf{r} \mid r > 0\}$ and $\phi = p \to \mathbf{0}$. Then any model $\mathcal{V}$ of T must satisfy $\mathcal{V}(p) = 0$, and hence $\|\phi\|_T = 1$. However, $T \nvdash p \to \mathbf{0}$: if this was the case, then $p \to \mathbf{0}$ would be provable from a finite subtheory T_0 of T, but then, let $r_0 = \min\{r \mid p \to \mathbf{r} \in T_0\}$; then, there is a model $\mathcal{V}$ of T_0 with $\mathcal{V}(p) = r_0 > 0$. Moreover, $|\phi|_T = 0$ since if $T \models \mathbf{s} \to (p \to \mathbf{0})$ for some s, then it holds that $T \models p \to (\mathbf{s} \to \mathbf{0})$ (by counterpositive) and thus $T \vdash p \to \mathbf{0}$ by the definition of the Goguen implication.

In fact, in the product logic it is not even possible to express formulas that must be interpreted in an intermediate truth value $a, 0 < a < 1$.

Lemma 3.18. *Let T be a theory over Π, then the following two properties hold:*

1. *if $T \cup \{\phi\}$ is inconsistent, then $T \vdash \neg\phi$,*
2. *if T is consistent, then for every ϕ at least one of $T \cup \{\phi\}, T \cup \{\neg\phi\}$ is consistent.*

Proof. 1. if $T \cup \{\phi\}$ is inconsistent, then for some $n, T \vdash \phi^n \to \mathbf{0}$, but then $T \vdash \phi \to \mathbf{0}$ (See exercise below)

2. assume that both theories are inconsistent, then $T \vdash \neg\phi$ and $T \vdash \neg\neg\phi$. then we have that $T \vdash \neg\phi \& \neg\neg\phi$, and thus $T \vdash \mathbf{0}$.

□

Exercise. Finish the proof of 1. by showing that $\Pi \vdash (\phi \& \phi \to \mathbf{0}) \to (\phi \to \mathbf{0})$.

Corollary 3.19. *For every consistent theory T over Π and every formula ϕ, there is a model $\mathcal{V}$ of T such that $\mathcal{V}(\phi) \in \{0, 1\}$.*

4 Wrap Up

We have studied a family of logics capable of dealing with different degrees of truth. All these logics, whose semantics are based on t-norms, share some characteristics that are summarized through the basic logic BL. However, the specific properties given by every different t-norm are well worth studying. For instance, every t-norm can express also the Gödel t-norm min, but its residuum is not so easy to simulate. Łukasiewicz logic can be easily extended for dealing with other truth constants; a property that is not shared by neither Gödel nor product logic.

Depending on the application in hand, some of these properties may be more useful than others. In a classical approach to logic, one may desire that conjunction is idempotent (Gödel logic); but non-idempotent logics may be useful for giving emphasis to some concepts (i.e., when one repeats a property, one is actually stating that it is satisfied to a high degree). Involutive negation is also an intuitive property, only satisfied by Łukasiewicz semantics.

If logic is used for knowledge representation, or in general for AI applications, one has to keep these variations in mind, when chosing the adequate semantics to be considered. These also influence the viability and complexity of reasoning within the formalism.

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