Fuzzy Logic

Exercise Sheet 3

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Exercise 1

Using only modus ponens and the axioms (A1)-(A8) prove the following formulas in BL.

a) $(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$

b) $\varphi \to \neg\neg\varphi$

Exercise 2

Remember that, as an algebraic structure, a lattice is defined as a triple $(L, \wedge, \vee)$, where L is a non-empty set and $\wedge$ and $\vee$ are binary operators that are associative, commutative and absorbing[1]. On the set L the binary relation $\leq$ is defined as

$$a \leq b \text{ iff } a \wedge b = a.$$

Prove that

a) The relation $\leq$ is an order relation, i.e. it is reflexive, transitive and antisymmetric.

b) The infimum of a and b with respect to $\leq$ is $a \wedge b$.

c) The supremum of a and b with respect to $\leq$ is $a \vee b$.

Exercise 3

Check whether the following structures are residuated lattices.

a) $\{\{0,1\}^n, \wedge, \vee, \wedge, \to, 0^n, 1^n\}$ for some fixed natural number n where $\vee$, $\wedge$, $\to$ denote the boolean operators taken pointwise.

b) $\{2^M, \cup, \cap, \Delta, /, M, \emptyset\}$ where M is a finite set, $\cup$, $\cap$ and Δ denote union, intersection and symmetric difference, respectively, and / is defined as $A / C := C \setminus A$. Hint: Notice the sequence in which $\cap$ and $\cup$ appear in the tuple. How does this affect the order relation defined in Exercise 2?

[1] Operators $\vee$ and $\wedge$ are called absorbing if they satisfy $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$

Exercise 4

Show that every divisible linearly ordered residuated lattice is a BL-algebra. Hint: use Lemma 2.26.