Automata and Logic

Exercise Sheet 5

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Exercise 23

Complete the proof of Lemma 2.2 from the lecture by showing that the class $(B_0)_\Sigma$ is closed under union.

Exercise 24

Let $(S, \circ)$ be a finite semigroup, $m \in S$, and $i, k, \ell \in \mathbb{N} \setminus \{0\}$ defined as in the proof of Thm. 2.4. Show that if k is minimal with the property described in the proof, then

$$(\{m^i, \dots, m^{i+k-1}\}, \circ, m^\ell)$$

is a group. Is $(\{m^i, \dots, m^{i+k-1}\}, \circ, m^\ell)$ still a group if k is not minimal?

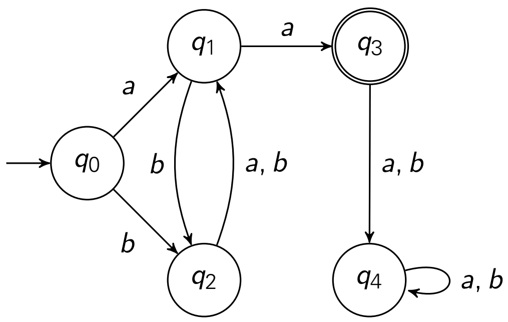
Exercise 25

Let $\Sigma := \{a, b\}$ and L_1, L_2 be the languages accepted by the automata displayed below. *Use the proof of Cor. 2.10* from the lecture to show that $L_1 \notin (B_0)_\Sigma$ and $L_2 \in (B_0)_\Sigma$.

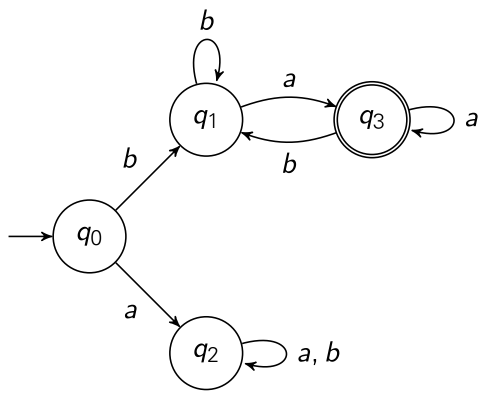
Moreover, represent L_2 as a Boolean combination of languages from the set

$$\{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^* u \mid u \in \Sigma^*\}.$$

(1)



(2)



Exercise 26

Prove or refute the following:

a) For every alphabet Σ and word $w \in \Sigma^*$, we have $\{w\} \in (B_0)_\Sigma$.

b) For every two alphabets Σ and Σ' with $\Sigma \subseteq \Sigma'$, and every language $L \subseteq \Sigma^*$, we have: if $L \in (B_0)_\Sigma$, then $L \in (B_0)_{\Sigma'}$.

c) Let $(M, \circ, 1)$ be a monoid, where 1 is the only idempotent element of M. Then $(M, \circ, 1)$ is a group.

d) Let $(S, \circ)$ be a semigroup with $e \in S$ being idempotent. Then $(eSe, \circ, e)$ is the largest submonoid of S with e as unit element.

e) Let $(S, \circ) \in \widehat{\mathbb{D}}$. If there exists an element $s \in S$ such that $(S, \circ, s)$ is a monoid, then $|S| = 1$.

Exercise 27

Let V be the class of all finite semigroups S such that for all idempotent elements $e \in S$, we have $Se = e$. Show that V is an S-variety ultimately defined by

$$yx^n = x^n \qquad (n \geq 1).$$

Exercise 28

Let $\Sigma := \{a, b, c, d\}$.

a) For $L \subseteq \Sigma^*$ with

$$\begin{aligned} L := \{&w \in \Sigma^* \mid w \text{ starts with } a \text{ or } b\} \cap \\ &\{w \in \Sigma^* \mid |w| \geq 3 \text{ and } w \text{ starts and ends with the same symbol}\}, \end{aligned}$$

give a quantifier-free formula ϕ using the signature $\{Q_a, Q_b, Q_c, Q_d, <, \min, \max, s, p\}$ such that $L(\phi) = L$.

b) Let

$$\phi := \neg(\neg Q_a(s(s(p(s(\min))))) \vee (s(\min) < p(p(\max)))).$$

Use the method described in the proof of Prop. 2.11 to describe $L(\phi)$ as a Boolean combination of languages from the set $\{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^* u \mid u \in \Sigma^*\}$.