Automata and Logic

Exercise Sheet 5

Prof. Dr.-Ing. Franz Baader
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Exercise 23

Complete the proof of Lemma 2.2 from the lecture by showing that the class \((B_0)_{\Sigma}\) is closed under union.

Exercise 24

Let \((S, \circ)\) be a finite semigroup, \(m \in S\), and \(i, k, \ell \in \mathbb{N} \setminus \{0\}\) defined as in the proof of Thm. 2.4. Show that if \(k\) is minimal with the property described in the proof, then

\[
(\{m^i, \ldots, m^i+k-1\}, \circ, m^\ell)
\]

is a group. Is \((\{m^i, \ldots, m^i+k-1\}, \circ, m^\ell)\) still a group if \(k\) is not minimal?

Exercise 25

Let \(\Sigma := \{a, b\}\) and \(L_1, L_2\) be the languages accepted by the automata displayed below. Use the proof of Cor. 2.10 from the lecture to show that \(L_1 \notin (B_0)_{\Sigma}\) and \(L_2 \in (B_0)_{\Sigma}\).

Moreover, represent \(L_2\) as a Boolean combination of languages from the set

\[
\{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^*u \mid u \in \Sigma^*\}.
\]
Exercise 26

Prove or refute the following:

a) For every alphabet \( \Sigma \) and word \( w \in \Sigma^* \), we have \( \{w\} \in (B_0)_\Sigma \).

b) For every two alphabets \( \Sigma \) and \( \Sigma' \) with \( \Sigma \subseteq \Sigma' \), and every language \( L \subseteq \Sigma^* \), we have: if \( L \in (B_0)_\Sigma \), then \( L \in (B_0)_{\Sigma'} \).

c) Let \((M, \circ, 1)\) be a monoid, where 1 is the only idempotent element of \( M \). Then \((M, \circ, 1)\) is a group.

d) Let \((S, \circ)\) be a semigroup with \( e \in S \) being idempotent. Then \((eSe, \circ, e)\) is the largest submonoid of \( S \) with \( e \) as unit element.

e) Let \((S, \circ)\) be a semigroup with \( e \in S \) being idempotent. If there exists an element \( s \in S \) such that \((S, \circ, s)\) is a monoid, then \(|S| = 1\).

Exercise 27

Let \( V \) be the class of all finite semigroups \( S \) such that for all idempotent elements \( e \in S \), we have \( Se = e \). Show that \( V \) is an \( S \)-variety ultimately defined by
\[
xy^n = x^n \quad (n \geq 1).
\]

Exercise 28

Let \( \Sigma := \{a, b, c, d\} \).

a) For \( L \subseteq \Sigma^* \) with
\[
L := \{w \in \Sigma^* \mid w \text{ starts with } a \text{ or } b\} \cap \\
\{w \in \Sigma^* \mid |w| \geq 3 \text{ and } w \text{ starts and ends with the same symbol}\},
\]
give a quantifier-free formula \( \phi \) using the signature \( \{Q_a, Q_b, Q_c, Q_d, <, \min, \max, s, p\} \) such that \( L(\phi) = L \).

b) Let
\[
\phi := \neg(\neg Q_a(s(s(p(s(\min)))))) \lor (s(\min) < p(p(\max)))).
\]
Use the method described in the proof of Prop. 2.11 to describe \( L(\phi) \) as a Boolean combination of languages from the set \( \{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^*u \mid u \in \Sigma^*\} \).