Automata and Logic

Exercise Sheet 1
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Exercise 1
Let $\Sigma = \{a, b\}$ be an alphabet and $\alpha := a^* b^* + b a^*$ a regular expression over $\Sigma$. Give a regular expression $\beta$ for the the complement language of $\alpha$, i.e. $\beta$ describes the set of words over $\Sigma$ that are not expressed by $\alpha$.

Exercise 2
Let $A$ be a non-deterministic automaton $A := (\{q_0, q_1, q_2\}, \{a, b\}, \{q_0\}, \Delta, \{q_1, q_2\})$ with $\Delta$ given by the following transition system:

![Transition Diagram]

Apply the power-set construction to $A$ in order to obtain a deterministic automaton that accepts the same language as $A$.

Exercise 3
For a language $L \subseteq \Sigma^*$, the Nerode right congruence $\rho_L$ is defined as follows. For $u, v \in \Sigma^*$, we have:

$$u \rho_L v \iff \text{for all } w \in \Sigma^*, uw \in L \Leftrightarrow vw \in L.$$ 

Let $A_L := (Q_L, \Sigma, q_L, \delta_L, F_L)$ be a deterministic automaton where:

- $Q_L := \{[u]_{\rho_L} \mid u \in \Sigma^*\}$ where $[u]_{\rho_L} := \{v \in \Sigma^* \mid u v \in L\}$,
- $q_L := [\epsilon]_{\rho_L}$ where $\epsilon$ denotes the empty word,
- $\delta_L([u]_{\rho_L}, a) := [ua]_{\rho_L}$ for $u \in \Sigma^*$, $a \in \Sigma$,
- $F_L := \{[u]_{\rho_L} \mid u \in L\}$.

Show the following for regular languages $L$:

a) $A_L$ is well-defined.
b) \( \mathcal{A}_L \) is minimal (w.r.t. the number of states), i.e. for every deterministic automaton \( \mathcal{A} = (Q, \Sigma, q_0, \delta, F) \) with \( L(\mathcal{A}) = L \), we have \( |Q_L| \leq |Q| \).

**Exercise 4**

Let \( \mathcal{A} \) be the automaton that accepts words over the alphabet \( \Sigma := \{a, b\} \) described by the following transition system:

![Automaton Diagram]

Construct an automaton \( \mathcal{A}' \) such that \( L(\mathcal{A}') = L(\mathcal{A}) \) and \( \mathcal{A}' \) is minimal.

**Exercise 5**

Prove the following by giving a decision procedure:

a) The *emptiness problem* for regular languages is decidable.

b) The *inclusion problem* for regular languages is decidable.