Automata and Logic

Exercise Sheet 2
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Exercise 6
Prove that the language \( L := \{ a^n b^n \mid n \geq 0 \} \) is not regular using Nerode’s Theorem.

Exercise 7
Let \( M := \{ 1, m \} \) with \( 1 \neq m \). Determine all operations \( \circ \) such that \(( M, \circ, 1)\) is a monoid.

Exercise 8
Consider the monoid \(( \mathbb{Z}, +, 0)\) and the following relations on \( \mathbb{Z} \), where \( 3 \mid z \) denotes that \( z \) is divided by 3 without remainder.

- \( z_1 R_1 z_2 \) iff \( 3 \mid (z_1 - z_2) \);
- \( z_1 R_2 z_2 \) iff \( 3 \mid z_1 \) and \( 3 \mid z_2 \), or \( 3 \nmid z_1 \) and \( 3 \nmid z_2 \).

For each \( z \in \mathbb{Z} \), \([z]_i\) denotes the equivalence class of \( z \) w.r.t. the relation \( R_i \). We now define the monoids \(( M_i, \circ_i, 1_i)\) for \( i \in \{1, 2\} \) as follows:

- \( M_i := \{ [z]_i \mid z \in \mathbb{Z} \} \)
- \([z]_i \circ_i [z']_i := [z + z']_i\)
- \( 1_i := [0]_i\)

Prove the following:

a) \( R_1 \) and \( R_2 \) are both equivalence relations.

b) \( R_1 \) is a congruence relation, but \( R_2 \) is not.

c) \(( M_1, \circ_1, 1_1)\) is well-defined, but \(( M_2, \circ_2, 1_2)\) is not.

Exercise 9
Let \( \mathcal{A} = (Q, \Sigma, q_0, \delta, F) \) be a deterministic finite automaton. In the lecture, we defined the relations \( \sim_\mathcal{A}, \sim_0, \sim_1, \ldots \subseteq Q \times Q \) as follows:

- \( q \sim_\mathcal{A} q' \) iff \( L(A_q) = L(A_{q'}) \);
- \( q \sim_0 q' \) iff \( \{ q, q' \} \subseteq F \) or \( \{ q, q' \} \cap F = \emptyset \);
- \( q \sim_{i+1} q' \) iff \( q \sim_i q' \) and \( \delta(q, a) \sim_i \delta(q', a) \) for all \( a \in \Sigma \).
Prove that there exists an $n \in \mathbb{N}$ such that $\sim_n = \sim_A$.

**Exercise 10**

Consider the monoids $M_i := (\{1, a, b\}, \circ_i, 1)$, for $i \in \{1, 2\}$, where $\circ_1$ is given by the following table:

<table>
<thead>
<tr>
<th>$\circ_1$</th>
<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

and $x \circ_2 y := y \circ_1 x$ for all $x, y \in \{1, a, b\}$.

For each $i \in \{1, 2\}$, find a regular language $L_i \subseteq \{a, b\}^*$ such that $M_i$ is the syntactic monoid of $L_i$, or prove that no such language exists.

**Exercise 11**

Let $\Sigma$ be an alphabet and $(M, \circ, 1)$ a monoid. Prove that every function $f : \Sigma \to M$ can be uniquely extended to a (monoid) homomorphism $\Phi : \Sigma^* \to M$. 