Automata and Logic
Exercise Sheet 4
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Exercise 19
Let $V$ be the $M$-variety of all commutative finite groups. Show that there exists a language $L \subseteq \{a\}^*$ such that $L \in L(V)_{\{a\}}$ but $L \notin L(V)_{\{a, b\}}$.

Exercise 20
Prove or refute the following: There is a language $L \subseteq \{a, b\}^*$ such that its syntactic semigroup $S_L$ and its syntactic monoid $M_L$ are isomorphic.

Exercise 21
For each of the following words over the alphabet $\{0, 1\}^k$, give a corresponding interpretation over the predicate symbols $P_1, \ldots, P_k$ as discussed in the lecture:

- $k = 2$: $(1, 1), (1, 0), (0, 1), (0, 0)$
- $k = 3$: $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)$
- $k = 3$: $(1, 1, 0), (1, 0, 1), (1, 1, 1), (1, 1, 0)$

Describe all interpretations that correspond to words of the language $L(((0, 1) \cdot (1, 0))^+) \subseteq (\{0, 1\}^2)^+$.

Exercise 22
Let $\Sigma = \{a, b\}$. For each of the following regular expressions $r_i$, give a first-order formula $\phi_i$ such that $L(r_i) = L(\phi_i)$.

- a) $r_1 = \Sigma^*$,
- b) $r_2 = \varepsilon$,
- c) $r_3 = (abb^*)^*$,
- d) $r_4 = a^*b^* + b^*a^*$, and
- e) $r_5 = (aaa \cdot \Sigma^*) + b^*$.

Exercise 23
Complete the proof of Lemma 2.2 from the lecture by showing that the class $(B_0)_\Sigma$ is closed under union.
Exercise 24

Let \((S, \circ)\) be a finite semigroup, \(m \in S\), and \(i, k, \ell \in \mathbb{N} \setminus \{0\}\) defined as in the proof of Thm. 2.4. Show that if \(k\) is minimal with the property described in the proof, then

\[\{(m^i, \ldots, m^{i+k-1}), \circ, m^\ell\}\]

is a group. Is \(\{(m^i, \ldots, m^{i+k-1}), \circ, m^\ell\}\) still a group if \(k\) is not minimal?

Exercise 25

Let \(\Sigma := \{a, b\}\) and \(L_1, L_2\) be the languages accepted by the automata displayed below. Use the proof of Cor. 2.10 from the lecture to show that \(L_1 \notin (B_0)_\Sigma\) and \(L_2 \in (B_0)_\Sigma\).

Moreover, represent \(L_2\) as a Boolean combination of languages from the set

\[\{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^* u \mid u \in \Sigma^*\}.

Exercise 26

Prove or refute the following:

a) For every alphabet \(\Sigma\) and word \(w \in \Sigma^*\), we have \(\{w\} \in (B_0)_\Sigma\).

b) For every two alphabets \(\Sigma\) and \(\Sigma'\) with \(\Sigma \subseteq \Sigma'\), and every language \(L \subseteq \Sigma^*\), we have: if \(L \in (B_0)_\Sigma\), then \(L \in (B_0)_{\Sigma'}\).

c) Let \((M, \circ, 1)\) be a monoid, where 1 is the only idempotent element of \(M\). Then \((M, \circ, 1)\) is a group.

d) Let \((S, \circ)\) be a semigroup with \(e \in S\) being idempotent. Then \((eSe, \circ, e)\) is the largest submonoid of \(S\) with \(e\) as unit element.

e) Let \((S, \circ) \in \hat{D}\). If there exists an element \(s \in S\) such that \((S, \circ, s)\) is a monoid, then \(|S| = 1\).