Term Rewriting Systems

Lecture
Tuesdays and Thursdays 16:40 – 18:10
(Rafael Peñaloza)

Tutorial
Wednesdays 14:50 – 16:20
(Marcel Lippmann)

http://lat.inf.tu-dresden.de/teaching/ss2014/TRS/
Franz Baader and Tobias Nipkow

Term Rewriting and *all That*


http://www4.informatik.tu-muenchen.de/~nipkow/TRaAT/
Term Rewriting

Terms

Expressions built from variables, constant symbols, and function symbols

Example:

- variables $x, y$
- constant symbol $0$
- function symbols $s$ (unary) and $+$ (binary; infix notation)

$$0, \ x + s(0), \ s(s(0)) + 0$$

Rewriting

Rules that express how can one term be transformed into another
Examples

Rewriting as computation mechanism

rules are applied in **one direction**
computes normal forms

- closely related to **functional programming**
- for example: **symbolic differentiation**

Rewriting as deduction mechanism

rules are applied in **both directions**
defines equivalence classes of terms

- **equational theory** in automated deduction
- for example: **group theory**
Terms are arithmetic expressions built with the operations:

- \(+\) (binary function symbol)
- \(*\) (binary function symbol)

the indeterminates \(X, Y\) (constant symbols) and
the numbers \(0, 1\) (constant symbols).

Example: \(((X+X)\*Y)+1\)

Additionally, the unary function symbol \(D_X\): partial derivative w.r.t. \(X\)
Rules for Symbolic Differentiation

1. \( D_X(X) \rightarrow 1 \)
2. \( D_X(Y) \rightarrow 0 \)
3. \( D_X(u + v) \rightarrow D_X(u) + D_X(v) \)
4. \( D_X(u \ast v) \rightarrow (u \ast D_X(v)) + (v \ast D_X(u)) \)
Important Properties of Term Rewriting Systems

Termination

Is it always true that after finitely many rule applications, we reach an expression that cannot be rewritten anymore? (normal form)

For the rules (R1) – (R4) this is the case (every rewriting execution terminates)

How to prove this?

\[ D_X(u \ast v) \rightarrow (u \ast D_X(v)) + (v \ast D_X(u)) \]

Example

\[ u + v \rightarrow v + u \]

Is non-terminating. Leads to an infinite sequence of applications

\[ X + Y \rightarrow Y + X \rightarrow X + Y \rightarrow Y + X \rightarrow X + Y \rightarrow Y + X \rightarrow \ldots \]
Important Properties of Term Rewriting Systems

Confluence

If different rules can be applied to a given term $t$, leading to different terms $t_1$ and $t_2$, can $t_1$ and $t_2$ be joined?

is there a term $s$ that can be reached from $t_1$ and $t_2$ by rule applications?
Confluence

Rules (R1) – (R4) are confluent

How to prove this?

If we add the rule: 

\[(R5) \quad u + 0 \rightarrow u\]

to this system, we lose confluence

Confluence regained by adding the rule

\[D_X(0) \rightarrow 0\]
Group Theory

Consider

- $\circ$ a binary function symbol
- $i$ a unary function symbol
- $e$ a constant symbol, and
- $x, y, z$ variable symbols

The class of groups is defined by the identities

\[
\begin{align*}
(G1) & \quad (x \circ y) \circ z \approx x \circ (y \circ z) \quad (\circ \text{ is associative}) \\
(G2) & \quad e \circ x \approx x \quad (e \text{ is a left-unit}) \\
(G3) & \quad i(x) \circ x \approx e \quad (i \text{ is left-inverse})
\end{align*}
\]

Identities are rules that can be applied in both directions

Word Problem

Given identities $E$ and terms $s, t$, can $s$ be rewritten into $t$ using $E$?
We can prove that $i$ is also a right-inverse; i.e. $e$ can be rewritten to $x \circ i(x)$

\begin{align*}
\text{(G1)} & \quad (x \circ y) \circ z \approx x \circ (y \circ z) \\
\text{(G2)} & \quad e \circ x \approx x \\
\text{(G3)} & \quad i(x) \circ x \approx e
\end{align*}

\begin{align*}
\text{(G3)} & \quad i(x \circ i(x)) \circ (x \circ i(x)) \\
\text{(G2)} & \quad i(x \circ i(x)) \circ (x \circ (e \circ i(x))) \\
\text{(G3)} & \quad i(x \circ i(x)) \circ (x \circ ((i(x) \circ x) \circ i(x))) \\
\text{(G1)} & \quad i(x \circ i(x)) \circ ((x \circ (i(x) \circ x)) \circ i(x)) \\
\text{(G1)} & \quad i(x \circ i(x)) \circ (((x \circ i(x)) \circ x) \circ i(x)) \\
\text{(G1)} & \quad i(x \circ i(x)) \circ (((x \circ i(x)) \circ (x \circ i(x)))) \\
\text{(G1)} & \quad (i(x \circ i(x)) \circ (x \circ i(x))) \circ (x \circ i(x)) \\
\text{(G3)} & \quad e \circ (x \circ i(x)) \\
\text{(G2)} & \quad x \circ i(x)
\end{align*}
Word Problem

Given identities $E$ and terms $s$, $t$, can $s$ be rewritten into $t$ using $E$?

Idea

Solve the word problem using rewriting in one direction

$\hat{s} = \hat{t}$

Problems

- equivalent terms can have different normal forms
- normal forms may not exist

Termination and confluence ensure the existence and uniqueness of normal forms