# Term Rewriting Systems

## Exercise Sheet 1

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### Exercise 1

Consider the reduction system $(M, \to)$ with $M = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1, C_2, C_3, C_4, D, E\}$ and $\to \subseteq M \times M$:

- $A_1 \to B_1$, $A_1 \to B_2$, $A_2 \to B_1$, $A_2 \to B_2$, $A_3 \to B_3$, $A_4 \to B_3$,
- $B_1 \to C_1$, $B_2 \to C_2$, $B_2 \to C_3$, $B_3 \to C_1$, $B_3 \to C_2$, $B_3 \to C_3$, $B_3 \to C_4$,
- $C_3 \to E$, $C_4 \to E$, and
- $D \to C_4$.

Answer the following questions.

a) Which of the following properties are satisfied by $\to$? Justify your answer.

   i) finite
   ii) symmetric
   iii) antisymmetric
   iv) reflexive
   v) irreflexive
   vi) transitive

b) Describe the following *closures*:
   $\xrightarrow{=}$, $\xrightarrow{+}$, $\xrightarrow{*}$, and $\leftrightarrow$.

### Exercise 2

Let $\to$ be the *symbolic differentiation relation* introduced in the lecture.

a) Compute the *normal forms* of the following terms:

   i) $D_X(((X * X) * X) + (X * X))$, and
   ii) $D_X((X * Y) + (Y * Y))$.

b) Prove that $\to$ is *terminating*.

### Exercise 3

In the lecture, a *group* was defined by the following identities:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{G1}$$
$$e \circ x \approx x \tag{G2}$$
$$i(x) \circ x \approx e \tag{G3}$$

a) Prove that groups satisfy the property that $e$ is a right unit, i.e.

$$x \circ e \approx x \tag{G2'}$$

by showing that $x \circ e$ can be transformed to $x$ using the identities G1, G2 and G3.

b) Consider the following identity:

$$x \circ i(x) \approx e \tag{G3'}$$

Prove that G1, G2 and G3′ do not imply G2′.

**Hint:** Give a model of G1, G2 and G3′ in which G2′ does not hold; such a model exists with only two elements.

### Exercise 4

Consider the following identities:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{R1}$$
$$(x \circ y) \circ x \approx x \tag{R2}$$

Prove or refute whether the following identities are implied by R1 and R2.

a) $(x \circ x) \approx x$

b) $(x \circ y) \circ z \approx x \circ z$