

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Term Rewriting Systems

Exercise Sheet 2

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Exercise 5

Which of the closure operators defined in the lecture commute? Prove or refute the validity of each of the following equations:

a)
$$(\stackrel{+}{\rightarrow})^{=} = (\stackrel{=}{\rightarrow})^{+}$$

b)
$$(\rightarrow \cup \stackrel{-1}{\rightarrow})^+ = \stackrel{+}{\rightarrow} \cup (\stackrel{+}{\rightarrow})^{-1}$$

c) $(\stackrel{-1}{\to})^+ = (\stackrel{+}{\to})^{-1}$

Exercise 6

Consider the following reduction relations:

- Let *M* be a set and 2^M the power set of *M*. We define the reduction relation \rightarrow_M on 2^M as follows: $A \rightarrow_M B$ iff $B \subsetneq A$.
- We define the reduction relation $\rightarrow_{p,q}$ on the non-negative integers as follows: $n \rightarrow_{p,q} n - p$ iff n > p, and $n \rightarrow_{p,q} n - q$ iff n > q.

For each of the following properties, describe those sets M and those non-negative integers p, q such that \rightarrow_M and $\rightarrow_{p,q}$ satisfy the following property:

- a) terminating,
- b) Church-Rosser,
- c) normalising, and
- d) confluent.

Exercise 7

In the lecture, we defined the set $A := \mathbb{N} \setminus \{0, 1\}$ and the following 'divisibility' relation on A:

 $\rightarrow := \{ (m, n) \mid m > n \text{ and there is some } \ell \in A \text{ with } n \cdot \ell = m \}.$

Prove that $\stackrel{*}{\leftrightarrow} = A \times A$.

Exercise 8

Disprove the following claim:

If (M, \rightarrow) is a reduction system such that $x \rightarrow y$ is decidable, then the set $\{x \in M \mid x \text{ is reducible}\}$ is also decidable.

Hint: Define a reduction relation \rightarrow on the non-negative integers such that $n \rightarrow m$ is decidable, but the problem whether *n* is reducible is undecidable.

Exercise 9

Consider the reduction system \rightarrow_S from the lecture with $S := \{abb \rightarrow aa, a \rightarrow b\}$. (Note: $\rightarrow_S = \{(uabbv, uaav) \mid u, v \in \{a, b\}^*\} \cup \{(uav, ubv) \mid u, v \in \{a, b\}^*\}$)

- a) Decide whether:
 - ababb $\stackrel{*}{\rightarrow}_{S}$ bbb
 - $aabb \stackrel{*}{\leftrightarrow}_S aaaaa$
- b) Prove that \rightarrow_S is finitely branching.
- c) Prove that \rightarrow_S is terminating by defining a monotonic mapping $\varphi \colon \{a, b\}^* \to \mathbb{N}$, i.e. a mapping such that $u \to_S v$ implies $\varphi(u) > \varphi(v)$.