TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Term Rewriting Systems

Exercise Sheet 3

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Exercise 10

Let **ack** : $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Ackermann function, i.e.

$$\mathbf{ack}(n,m) = \begin{cases} m+1 & \text{if } n = 0, \\ \mathbf{ack}(n-1, 1) & \text{if } m = 0, \\ \mathbf{ack}(n-1, \mathbf{ack}(n, m-1)) & \text{otherwise.} \end{cases}$$

Choose an appropriate order on pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ and use well-founded induction to prove that the function **ack** is well-defined, i.e. for each pair of non-negative integers it determines a unique value.

Exercise 11

Prove Lemma 2.21 from the lecture: If $>_A$ is a strict order on A, and $>_B$ is a strict order on B, then the lexicographic product $>_{A \times B}$ is a strict order on $A \times B$.

Exercise 12

Prove that, for a strict order $\succ$, its induced multiset order $\succ_{\mathrm{mul}}$ is a strict order.

Exercise 13

Let $\succ$ be a strict, decidable order on a finite set A.

a) Show that the induced multiset order $\succ_{\mathrm{mul}}$ is decidable.

b) Design a simpler decision procedure for the case when $\succ$ is a total strict order on A.

Exercise 14

Let $(A, \succ)$ be a strict order. Prove or refute the following claims.

a) If A has a smallest element, then $\succ$ is well-founded.

b) If every non-empty subset of A has a smallest element, then $\succ$ is well-founded.

c) If $\succ$ is well-founded, then, for each element $a \in A$, there are only finitely many b with $a \succ b$.

d) If $\succ$ is well-founded, then, for each element $a \in A$, there is an $n_a \in \mathbb{N}$ such that each $\succ$-path starting at a is of length at most n_a.

e) If $\succ$ is well-founded, then each non-empty subset of A has a smallest element.