Term Rewriting Systems
Exercise Sheet 4
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Exercise 15
The reduction relation $\rightarrow$ enjoys the diamond property if
$$y_1 \leftarrow x \rightarrow y_2 \implies \exists z. y_1 \rightarrow z \leftarrow y_2.$$  
Prove that, if $\rightarrow$ enjoys the diamond property, then every element $x$ is either in normal form or does not have a normal form.

Exercise 16
Let $(A, \rightarrow_1 \cup \rightarrow_2)$ be the reduction system obtained from the reduction systems $(A, \rightarrow_1)$ and $(A, \rightarrow_2)$ by building the union of the two reduction relations.
Prove or refute: If $\rightarrow_1$ and $\rightarrow_2$ are confluent, then so is $\rightarrow_1 \cup \rightarrow_2$.

Exercise 17
Does strong confluence imply the following property?
$$y_1 \leftarrow x \rightarrow y_2 \implies \exists z. y_1 \rightarrow z \leftarrow y_2$$
Give a proof or counterexample.

Exercise 18
Consider the terms $s = f(x, g(h(k(k(y)), x), z), h(x, y))$ and $t = g(z, h(x, k(k(y))))$.
Describe $t|_1$, $t|_{1111}$, $t|_{11111}$, $t[s]_2$, and $t[s][21]$.

Exercise 19
Prove the second part of Lemma 3.4 by induction on the length of words denoting positions:
If $p \in \text{Pos}(s)$ and $q \in \text{Pos}(t)$, then
$$(s[t]_p)|_{pq} = t|_q$$
$$(s[t]_p)[r]_{pq} = s[t[r]_q]_p$$

Exercise 20
Let $E$ be a set of identities and $\rightarrow_E$ the induced reduction relation. Prove that $\rightarrow_E$ and $\rightarrow^*_E$ are rewrite relations, i.e. are closed under substitutions and compatible with $\Sigma$-operations.