TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Term Rewriting Systems

Exercise Sheet 8

Dr. rer. nat. Rafael Peñaloza / Marcel Lippmann
Summer Semester 2014

Exercise 32

The following problem is known as *Hilbert's 10th problem*, and it has been proved to be undecidable.
Given: One polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$.
Question: Are there $a_1, \ldots, a_n \in \mathbb{N}$ such that $P(a_1, \ldots, a_n) = 0$?

Show that the undecidability of Hilbert's 10th problem implies the undecidability of the following problem, which you know from the lecture:
Given: Two polynomials $P, Q \in \mathbb{N}[X_1, \ldots, X_n]$ and a decidable set $A \subseteq \mathbb{N} \setminus \{0\}$.
Question: Is $P >_A Q$? That is, is $P(a_1, \ldots, a_n) > Q(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$?

Exercise 33

Let $P \in \mathbb{Z}[X]$ be a polynomial with one indeterminate and coefficients in $\mathbb{Z}$.

   a) Prove for all $r \in \mathbb{Z}$: if $P(r) = 0$, then $r | a_0$, i.e. any root of P divides a_0.

   b) Devise a decision procedure, which for each polynomial $P \in \mathbb{Z}[X]$ decides whether P has a root in $\mathbb{Z}$.

   c) Show that for polynomials with more than one indeterminate, the roots need not satisfy such a property.

Exercise 34

Consider the TRS $R = \{g(x, g(y, z)) \to g(g(x, y), z), \ g(g(x, y), z) \to g(y, y)\}$. Use a polynomial interpretation to prove that R terminates.
Hint: Try $p_g = xy + y^2$.

Exercise 35

A TRS R is called *right-irreducible* if each r with $\ell \to r \in R$ is irreducible. Prove or refute the following claim: If R is right-ground and right-irreducible, then R is terminating.

Exercise 36

Prove Lemma 5.20 from the lecture: Let $\mathcal{A}$ be a monotone polynomial interpretation of Σ. Then $f^{\mathcal{A}}$ is a monotone function for each $f \in \Sigma$.

Exercise 37

Let R be a finite TRS. We use $|t|_f$ to denote the number of occurrences of the symbol $f \in \Sigma$ in the term t. Prove the following claim: There is a positive integer k_R such that $s \to_R t$ implies $|t|_f \leq k_R(|s|_f + 1)$ for all terms s, t.

Exercise 38

Let R_{Ack} be the TRS from the lecture with

$$R_{\text{Ack}} = \{a(0, y) \to s(y), a(s(x), 0) \to a(x, s(0)), a(s(x), s(y)) \to a(x, a(s(x), y))\}.$$

Prove that there is no primitive recursive function that provides an upper bound for the length of reductions given the length of the first term. More precisely, prove that there is no primitive recursive function f such that $t_1 \to_{R_{\text{Ack}}} t_2 \to_{R_{\text{Ack}}} \cdots \to_{R_{\text{Ack}}} t_k$ implies $k \leq f(|t_1|)$.

Hint: Use the previous exercise and the fact the the Ackermann function grows faster than any primitive recursive function.