Exercise 12

Let \( \Sigma := \{a, b\} \), \( M := \{0, 1, 2\} \), and let \( \circ : M \times M \to M \) be defined as \( x \circ y := (x + y) \mod 3 \). We define mappings \( \Phi, \Phi' : \Sigma^* \to M \) by setting \( \Phi(w) := |w| \mod 3 \) and \( \Phi'(w) := |w|_a \mod 3 \), where \( |w| \) denotes the length of \( w \) and \( |w|_a \) the number of occurrences of the symbol \( a \) in \( w \).

a) Show that both \( \Phi \) and \( \Phi' \) are monoid homomorphisms from \((\Sigma^*, \cdot, \varepsilon)\) into \((M, \circ, 0)\).

b) For each of the languages \( \Phi^{-1}(\{0, 2\}) \), \( \Phi^{-1}(\{1\}) \) and \( (\Phi')^{-1}(\{1\}) \) devise a finite automaton that recognises the language.

Exercise 13

Let \( \Sigma \) be an alphabet, \( L \subseteq \Sigma^* \) a language, and \((M, \circ, 1)\) a monoid. Prove that \( L \) is accepted by \((M, \circ, 1)\), if and only if \( \overline{L} \) is also accepted by \((M, \circ, 1)\).

Exercise 14

Determine the syntactic monoid of the language described by \( a^* ba^* \).

Exercise 15

Let \( L \subseteq \Sigma^* \), and \( \approx \) be an equivalence relation on \( \Sigma^* \). Consider the following property:

For all \( u, v \in \Sigma^* \), if \( u \in L \) and \( u \approx v \), then \( v \in L \). \((\ast)\)

a) The proof of Corollary 1.13 from the lecture depends on the fact that the syntactical congruence \( \sim_L \) has property \((\ast)\). Prove this.

b) Show that \( \sim_L \) is the coarsest congruence relation with property \((\ast)\).

c) Show that the Nerode right congruence \( \rho_L \) is the coarsest right congruence with property \((\ast)\).

Note: An equivalence relation \( \approx_2 \) is coarser than \( \approx_1 \) if for every \( x, y \), \( x \approx_1 y \) implies \( x \approx_2 y \). (In particular, \( \approx_2 \) has at most as many equivalence classes as \( \approx_1 \).)

Exercise 16

Show that any submonoid of a finite group is also a group.
Exercise 17
Let $V$ be an $M$-variety. Show that $L(V)_\Sigma$ is closed under union without using Thm. 1.22 from the lecture.

Exercise 18
Let $\Sigma$ be an alphabet. Prove or refute the following claims:

a) Every regular language $L \subseteq \Sigma^*$ is accepted by its syntactic monoid.

b) If $L \subseteq \Sigma^*$ is accepted by a finite group, then the syntactic monoid of $L$ is a finite group.

c) For every regular language $L \subseteq \Sigma^*$, the syntactic monoid $M_L$ is the smallest monoid accepting $L$; i.e. for every monoid $M$ accepting $L$, we have $|M_L| \leq |M|$.

d) For a word $w = a_1 \ldots a_n$, let $\overline{w}$ denote the mirror image of $w$, i.e. $\overline{w} = a_n \ldots a_1$. For a language $L \subseteq \Sigma^*$, we define $\overline{L} := \{ \overline{w} \mid w \in L \}$. **Claim**: If the minimal automaton for $L$ has $n$ states, then the minimal automaton for $\overline{L}$ has also $n$ states.

Exercise 19
Let $L_1$ be the language over $\{a\}$ described by $a^+$, and let $L_2$ be the language over $\{a, b\}$ described by $(a + b)^*b(a + b)^*$.

a) Is there a monoid that accepts both $L_1$ and $L_2$?

b) Are the syntactic monoids of those languages isomorphic?

Exercise 20
Let $L_1$ and $L_2$ be two languages over the same alphabet $\Sigma$ that are accepted by the same monoid $(M, \circ, 1)$. Prove or refute the following statements:

a) $M$ accepts $L_1 \cap L_2$.

b) $M$ accepts $L_1 \cup L_2$.

c) $M$ accepts $L_1 \cdot L_2$. 