TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Automata and Logic

Exercise Sheet 3

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Exercise 12

Let $\Sigma := \{a, b\}$, $M := \{0, 1, 2\}$, and let $\circ\colon M \times M \to M$ be defined as $x \circ y := (x + y) \bmod 3$. We define mappings $\Phi, \Phi'\colon \Sigma^* \to M$ by setting $\Phi(w) := |w| \bmod 3$ and $\Phi'(w) := |w|_a \bmod 3$, where $|w|$ denotes the *length* of w and $|w|_a$ the number of occurrences of the symbol a in w.

a) Show that both Φ and Φ' are monoid homomorphisms from $(\Sigma^*, \cdot, \varepsilon)$ into $(M, \circ, 0)$.

b) For each of the languages $\Phi^{-1}(\{0, 2\})$, $\Phi^{-1}(\{1\})$ and $(\Phi')^{-1}(\{1\})$ devise a finite automaton that recognises the language.

Exercise 13

Let Σ be an alphabet, $L \subseteq \Sigma^*$ a language, and $(M, \circ, 1)$ a monoid. Prove that L is accepted by $(M, \circ, 1)$, if and only if $\overline{L}$ is also accepted by $(M, \circ, 1)$.

Exercise 14

Determine the syntactic monoid of the language described by a^*ba^*.

Exercise 15

Let $L \subseteq \Sigma^*$, and $\approx$ be an equivalence relation on Σ^*. Consider the following property:

$$\text{For all } u, v \in \Sigma^*, \text{ if } u \in L \text{ and } u \approx v, \text{ then } v \in L. \tag{$*$}$$

a) The proof of Corollary 1.13 from the lecture depends on the fact that the syntactical congruence $\sim_L$ has property $(*)$. Prove this.

b) Show that $\sim_L$ is the coarsest congruence relation with property $(*)$.

c) Show that the Nerode right congruence ρ_L is the coarsest right congruence with property $(*)$.

Note: An equivalence relation $\approx_2$ is *coarser* than $\approx_1$ if for every x, y, $x \approx_1 y$ implies $x \approx_2 y$. (In particular, $\approx_2$ has at most as many equivalence classes as $\approx_1$.)

Exercise 16

Show that any submonoid of a finite group is also a group.

Exercise 17

Let V be an M-variety. Show that $L(V)_\Sigma$ is closed under union *without* using Thm. 1.22 from the lecture.

Exercise 18

Let Σ be an alphabet. Prove or refute the following claims:

a) Every regular language $L \subseteq \Sigma^*$ is accepted by its syntactic monoid.

b) If $L \subseteq \Sigma^*$ is accepted by a finite group, then the syntactic monoid of L is a finite group.

c) For every regular language $L \subseteq \Sigma^*$, the syntactic monoid M_L is the smallest monoid accepting L; i.e. for every monoid M accepting L, we have $|M_L| \leq |M|$.

d) For a word $w = a_1 \dots a_n$, let $\overleftarrow{w}$ denote the mirror image of w, i.e. $\overleftarrow{w} = a_n \dots a_1$. For a language $L \subseteq \Sigma^*$, we define $\overleftarrow{L} := \{\overleftarrow{w} \mid w \in L\}$. **Claim:** If the minimal automaton for L has n states, then the minimal automaton for $\overleftarrow{L}$ has also n states.

Exercise 19

Let L_1 be the language over $\{a\}$ described by a^+, and let L_2 be the language over $\{a, b\}$ described by $(a+b)^*b(a+b)^*$.

a) Is there a monoid that accepts both L_1 and L_2?

b) Are the syntactic monoids of those languages isomorphic?

Exercise 20

Let L_1 and L_2 be two languages over the same alphabet Σ that are accepted by the same monoid $(M, \circ, 1)$. Prove or refute the following statements:

a) M accepts $L_1 \cap L_2$.

b) M accepts $L_1 \cup L_2$.

c) M accepts $L_1 \cdot L_2$.