Automata and Logic

Exercise Sheet 5

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Exercise 29

Let V be the class of all finite semigroups S such that for all idempotent elements $e \in S$, we have $Se = e$. Show that V is an S-variety ultimately defined by

$$yx^n = x^n \qquad (n \geq 1).$$

Exercise 30

Let $\Sigma := \{a, b, c, d\}$.

a) For $L \subseteq \Sigma^*$ with

$$\begin{aligned} L := \{w \in \Sigma^* \mid w \text{ starts with } a \text{ or } b\} \cap \\ \{w \in \Sigma^* \mid |w| \geq 3 \text{ and } w \text{ starts and ends with the same symbol}\}, \end{aligned}$$

give a quantifier-free formula ϕ using the signature $\{Q_a, Q_b, Q_c, Q_d, <, \min, \max, s, p\}$ such that $L(\phi) = L$.

b) Let

$$\phi := \neg(\neg Q_a(s(s(p(s(\min))))) \vee (s(\min) < p(p(\max)))).$$

Use the method described in the proof of Prop. 2.11 to describe $L(\phi)$ as a Boolean combination of languages from the set $\{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^* u \mid u \in \Sigma^*\}$.

Exercise 31

Let Σ be an alphabet. A language $L \subseteq \Sigma^*$ is called *definite* for Σ if there exists an $n \in \mathbb{N}$ such that we have for all $w \in L$:

$$\text{if } w = uv \text{ with } |u| = n \text{ then } u\Sigma^* \subseteq L.$$

Show that $L \subseteq \Sigma^*$ is definite for Σ iff L is a Boolean combination of languages of the form $\{w\Sigma^* \mid w \in \Sigma^*\}$.

Exercise 32

Let $\Sigma := \{0,1\}^k$. Show that the following statements are equivalent:

- L is definite for Σ.
- There exists a quantifier-free closed first-order formula ϕ over the signature $\{P_1, \dots, P_k, <, \mathrm{min}, s\}$ with $L(\phi) = L \setminus \{\varepsilon\}$.

Exercise 33

Let Σ, Γ be two alphabets, and let $L \subseteq \Sigma^*$. Prove or refute the following claims:

a) $L \in \mathrm{SF}_\Sigma \implies L \in \mathrm{SF}_{\Sigma \cup \Gamma}$

b) $L \in \mathrm{SF}_{\Sigma \cup \Gamma} \implies L \in \mathrm{SF}_\Sigma$

Exercise 34

For $\Sigma := \{a, b\}$, check whether the following languages are star-free:

a) $L_1 := (ab)^*$

b) $L_2 := \{w \mid |w|_a = 3k \text{ for some } k \in \mathbb{N}\}$

c) $L_3 := a(aba)^*b$

Use Thm. 3.6 from the lecture or give a star-free description of the language.