Remark: Sometimes it is more appropriate to consider semigroups instead of monoids.

Semigroup: has only a binary associative operation (no unit element required).

Syntactic Semigroup: \( \nu_L \) is also a congruence on the free semigroup \( \Sigma^+ = \Sigma^* \setminus \{ \varepsilon \} \). For \( L \subseteq \Sigma^* \) the syntactic semigroup is \( \Sigma^+/\nu_L \).

Alternatively: The syntactic semigroup is the transition semigroup \( \{ \delta_w \mid w \in \Sigma^+ \} \) of the minimal automaton of \( L \).

S-Variety: Class of finite semigroups closed under direct products, homomorphic images, and building sub-semigroups.

Note: even if \( L \subseteq S \), the unit element need not be an element of the sub-semigroups of \( S \).
EQUATIONS: S-varieties can also ultimately be defined by equations (Prop. 1.20 holds in a semigroup variant 1.20s).
These equations may not contain 1.

NOTE: S-varieties sometimes yield a more fine-grained division into classes.
Consider the equation \( x \cdot y = y \).
Only trivial monoids satisfy this equation: for \( w \in M \) we have
\[
M = 1_M \cdot w \quad w = 1_M.
\]
But non-trivial semigroups satisfy \( x \cdot y = y \):

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is associative

CLASS OF LANGUAGES: If \( V \) is an S-variety, then
\[
L(V)_\Sigma = \{ L \subseteq \Sigma^* \mid S_L \in V \}
\]

Lem. 1.21 and Prop 1.22 also hold in a semigroup variant 1.21s and 1.22s.