Introduction to Automatic Structures

Summer Semester 2016

Exercise Sheet 3 – Word Automatic Structures and Logic

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Exercise 3.1 Let $f\colon \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be a bijection, and consider the structure

$$\mathcal{A}_f := (\{0,1\}^*; \mathsf{graph}(f)) \quad \text{where} \quad \mathsf{graph}(f) := \{\, (u,v,f(u,v)) \mid u,v \in \{0,1\}^* \,\}.$$

Use the Constant Growth Lemma to prove that $\mathcal{A}_f$ is not automatic.

Exercise 3.2 Let R be a n-ary relation on Σ^*. We define the following notions:

- R is *locally bounded* if there is a $k \geq 0$ such that for all $x_1, \ldots, x_{n-1} \in \Sigma^*$, there are at most k words $x_n \in \Sigma^*$ such that $(x_1, \ldots, x_n) \in R$.
- R is *locally finite* if for all $x_1, \ldots, x_{n-1} \in \Sigma^*$, there are only finitely many words $x_n \in \Sigma^*$ such that $(x_1, \ldots, x_n) \in R$.

Prove or refute the following generalizations of the Constant Growth Lemma:

(a) If R is automatic and locally bounded, then there is a constant $c \in \mathbb{N}$ such that for all $(x_1, \ldots, x_n) \in R$ the inequality $|x_n| \leq \max\{|x_1|, \ldots, |x_{n-1}|\} + c$ is satisfied.

(b) If R is automatic and locally finite, then there is a constant $c \in \mathbb{N}$ such that for all $(x_1, \ldots, x_n) \in R$ the inequality $|x_n| \leq \max\{|x_1|, \ldots, |x_{n-1}|\} + c$ is satisfied.

(c) If R is automatic, then there is a constant $c \in \mathbb{N}$ such that for all $(x_1, \ldots, x_n) \in R$ the inequality $|x_n| \leq \max\{|x_1|, \ldots, |x_{n-1}|\} + c$ is satisfied.

Exercise 3.3 Let Σ be an alphabet with a total order $\leq$.

(a) The *lexicographic order* $\leq_{\mathsf{lex}}$ on Σ^* is defined by

$$a_1 \ldots a_m \leq_{\mathsf{lex}} b_1 \ldots b_n :\Leftrightarrow m = 0, \text{ or } a_1 < b_1, \text{ or } a_1 = b_1 \text{ and } a_2 \ldots a_m \leq_{\mathsf{lex}} b_2 \ldots b_n.$$

Show that $(\Sigma^*; \leq_{\mathsf{lex}})$ is automatic.

(b) The *length-lexicographic order* $\leq_{\mathsf{llex}}$ is a binary relation on Σ^* where

$$w_1 \leq_{\mathsf{llex}} w_2 :\Leftrightarrow \mathsf{length}(w_1) < \mathsf{length}(w_2), \text{ or } \mathsf{length}(w_1) = \mathsf{length}(w_2) \text{ and } w_1 \leq_{\mathsf{lex}} w_2.$$

Show that $(\Sigma^*; \leq_{\mathsf{llex}})$ is automatic.

Exercise 3.4 Complete the construction of the automaton M_α in the proof of Theorem 3.1 for the remaining cases: equality, negation, and disjunction.

Exercise 3.5 Let $\mathcal{A}$ be a structure. Which of the following statements are true? Justify your answer.

(a) If model checking is decidable for $\mathcal{A}$, then the FOL-theory of $\mathcal{A}$ is also decidable.

(b) If there is an algorithm for query evaluation for $\mathcal{A}$, then there is also an algorithm that decides model checking for $\mathcal{A}$.

(c) If the FOL-theory of $\mathcal{A}$ is decidable, then model checking for $\mathcal{A}$ is also decidable.

Exercise 3.6 Consider the structure

$$\mathcal{A} = (\{0,1\}^* ; \preceq, S_0, S_1, \mathsf{EqualLength})$$

from Example 2.5, where $\preceq$ is the prefix relation, S_0 and S_1 append 0 and 1, respectively, and $\mathsf{EqualLength}$ checks for equal length. For each of the following relations R_i, give a FOL-formula ϕ_i such that $(\mathcal{A}, \bar{a}) \models \phi_i(\bar{x})$ if, and only if, $\bar{a} \in R_i$.

(a) $R_1 := \{\, (u,v) \in (\{0,1\}^*)^2 \mid \mathsf{length}(u) \leq \mathsf{length}(v) \,\}$

(b) $R_2 := \{\, (u,v) \in (\{0,1\}^*)^2 \mid \text{the } |v|\text{-th symbol in } u \text{ is } 0 \,\}$

(c) $R_3 := \{\, (u,v,w) \in (\{0,1\}^*)^3 \mid u \text{ and } v \text{ differ in the } |w|\text{-th symbol} \,\}$

(d) $R_4 := \{\, (u,v,w) \in (\{0,1\}^*)^3 \mid u \text{ is the longest common prefix of } v \text{ and } w \,\}$

Exercise 3.7 Let $(L; \leq)$ be a poset.

(a) Give a FOX-formula to describe the pairs (x,y) such that the interval $[x,y]$ contains an even number of elements.

(b) Give a FO-formula that is valid if, and only if, $(L; \leq)$ is a tree.

(c) Define a FO$^\infty$-formula that describes those elements in $(L; \leq)$ with infinitely many lower neighbors.

(d) Define a FO-formula that characterizes elements having a supremum in $(L; \leq)$.

Hint. A supremum of two elements x and y is an element that is smallest w.r.t. the property of being greater than both x and y.