Exercise 4.1 Which of the following statements hold true? Justify your answer.

(a) If $A$ is a universal automatic structure, and $B$ is isomorphic to $A$, then also $B$ is universal and automatic.

(b) If $A$ is a universal automatic structure, and $A$ is FOL-definable in $B$, then also $B$ is universal and automatic.

(c) There are structures $A$, $B$, and $C$, such that $A$ is FOL-definable in $B$, and $B$ is FOL-definable in $C$, but $A$ is not FOL-definable in $C$.

(d) There is a universal automatic structure with finite domain.

(e) There are no universal automatic structures with only unary relations.

(f) For each Turing machine $M$ over the input alphabet $\Sigma$, there is an automatic structure $A_M$ such that the structure $(\Sigma^*; L(M))$, where $L(M) := \{ w \in \Sigma^* \mid M$ accepts $w \}$, is FOL-definable in $A_M$.

Exercise 4.2 The Post Correspondence Problem (PCP) is a well-known undecidable problem. The version we consider here is defined as follows. An instance $I := ((u_1, \ldots, u_n), (v_1, \ldots, v_n))$ of the PCP consists of two sequences of words over the alphabet $\{0, 1\}$. We call $s := s_1 \ldots s_k \in \{1, \ldots, n\}^+$ a solution of $I$ if

$$u_{s_1} \ldots u_{s_k} = v_{s_1} \ldots v_{s_k}.$$  

The sequence $((\epsilon, \epsilon), (u_{s_1}, v_{s_1}), \ldots, (u_{s_k} \ldots u_{s_k}, v_{s_1} \ldots v_{s_k}))$ is called the construction of $s$.

In this exercise we examine a connection between the PCP and the FOL-theory of the structure $A := (\{0, 1, \#, \$\}^*, \text{is}_\epsilon, \text{is}_0, \text{is}_1, \text{is}_\#, \text{is}_\$, \circ)$$

where $\text{is}_\epsilon, \text{is}_0, \text{is}_1, \text{is}_\#$, and $\text{is}_\$, are unary predicates that decide whether a word is the empty word, the word 0, 1, #, or $\$$, respectively, and the ternary relation $\circ$ checks whether in a given tuple $(w_1, w_2, w_3)$ the word $w_3$ is the concatenation of $w_1$ and $w_2$.

(a) Give FOL-formulae that define the following relations in $A$.

(i) The prefix relation $\preceq$, and the suffix relation $\succeq$.

(ii) The substring relation $\subseteq$. 

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(iii) The unary relation only$_{01}$ that contains all words from $\{0, 1\}^*$. 

(iv) For a given word $u \in \{0, 1, \#, $\}^*$, the unary relation $is_u$ that contains only the word $u$ itself.

(b) For every instance $I := ((u_1, \ldots, u_n), (v_1, \ldots, v_n))$ of the PCP, define a FOL-sentence $\phi_I$ with the following property: $I$ has a solution if, and only if, $\phi_I$ belongs to the FOL-theory of $A$.

$\text{Hint.}$ You can encode the construction of the solution $s$ as the string $S := \#\#\#\ldots\#\#u_1\$\#\ldots\#u_n\$\#\ldots\#v_1\$\ldots\#v_n\#.$

(c) Use your previous results to prove or refute the claim that $A$ has an automatic presentation.

Exercise 4.3 Show that the following structures are isomorphic to structures that are FOL-definable in the universal structure $(\{0, 1\}^*; \preceq, S_0, S_1, \text{EqualLength})$. Give the corresponding FOL-formulae.

(a) $(\mathbb{N}; \leq)$

(b) $(\mathbb{N}; +)$

The next Exercise 4 will not be discussed during the tutorial. Instead, solutions can be submitted via mail to francesco.kriegel@tu-dresden.de until 13th June 2016 – individual feedback will be provided.

Exercise 4.4 (Homework) (a) Give a universal structure other than $(\{0, 1\}^*; \preceq, S_0, S_1, \text{EqualLength})$.

(b) Show that an automatic structure over the unary alphabet can never be universal.