TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Introduction to Automatic Structures

Summer Semester 2016

Exercise Sheet 4 – Word Automatic Structures and First-Order Logic

24th May 2016

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Hint. Recall that an automatic structure $\mathcal{U}$ is *universal* if for each structure $\mathcal{A}$, there is an automatic presentation of $\mathcal{A}$ if, and only if, $\mathcal{A}$ has an automatic presentation that is FOL-definable in $\mathcal{U}$.

Exercise 4.1 Which of the following statements hold true? Justify your answer.

(a) If $\mathcal{A}$ is a universal automatic structure, and $\mathcal{B}$ is isomorphic to $\mathcal{A}$, then also $\mathcal{B}$ is universal and automatic.

(b) If $\mathcal{A}$ is a universal automatic structure, and $\mathcal{A}$ is FOL-definable in $\mathcal{B}$, then also $\mathcal{B}$ is universal and automatic.

(c) There are structures $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, such that $\mathcal{A}$ is FOL-definable in $\mathcal{B}$, and $\mathcal{B}$ is FOL-definable in $\mathcal{C}$, but $\mathcal{A}$ is not FOL-definable in $\mathcal{C}$.

(d) There is a universal automatic structure with finite domain.

(e) There are no universal automatic structures with only unary relations.

(f) For each Turing machine M over the input alphabet Σ, there is an automatic structure $\mathcal{A}_M$ such that the structure $(\Sigma^*; L(M))$, where $L(M) := \{ w \in \Sigma^* \mid M \text{ accepts } w \}$, is FOL-definable in $\mathcal{A}_M$.

Exercise 4.2 The *Post Correspondence Problem (PCP)* is a well-known undecidable problem. The version we consider here is defined as follows. An *instance* $I := ((u_1, \ldots, u_n), (v_1, \ldots, v_n))$ of the PCP consists of two sequences of words over the alphabet $\{0, 1\}$. We call $s := s_1 \ldots s_k \in \{1, \ldots, n\}^+$ a *solution* of I if

$$u_{s_1} \ldots u_{s_k} = v_{s_1} \ldots v_{s_k}.$$

The sequence $((\epsilon, \epsilon), (u_{s_1}, v_{s_1}), \ldots, (u_{s_1} \ldots u_{s_k}, v_{s_1} \ldots v_{s_k}))$ is called the *construction* of s.

In this exercise we examine a connection between the PCP and the FOL-theory of the structure

$$\mathcal{A} := (\{0, 1, \#, \$\}^*, \mathsf{is}_\epsilon, \mathsf{is}_0, \mathsf{is}_1, \mathsf{is}_\#, \mathsf{is}_\$, \circ)$$

where $\mathsf{is}_\epsilon, \mathsf{is}_0, \mathsf{is}_1, \mathsf{is}_\#$, and $\mathsf{is}_\$$, are unary predicates that decide whether a word is the empty word, the word 0, 1, #, or $, respectively, and the ternary relation $\circ$ checks whether in a given tuple (w_1, w_2, w_3) the word w_3 is the concatenation of w_1 and w_2.

(a) Give FOL-formulae that define the following relations in $\mathcal{A}$.

  (i) The prefix relation $\preceq$, and the suffix relation $\succeq$.

  (ii) The substring relation $\subseteq$.

(iii) The unary relation only_{01} that contains all words from $\{0,1\}^*$.

(iv) For a given word $u \in \{0,1,\#,\$\}^*$, the unary relation is_u that contains only the word u itself.

(b) For every instance $I := ((u_1,\ldots,u_n),(v_1,\ldots,v_n))$ of the PCP, define a FOL-sentence ϕ_I with the following property: I has a solution if, and only if, ϕ_I belongs to the FOL-theory of $\mathcal{A}$.

Hint. You can encode the construction of the solution s as the string

$$s := \#\$\#u_{s_1}\$v_{s_1}\#u_{s_1}u_{s_2}\$v_{s_1}v_{s_2}\#\cdots\#u_{s_1}u_{s_2}\cdots u_{s_k}\$v_{s_1}v_{s_2}\cdots v_{s_k}\#.$$

(c) Use your previous results to prove or refute the claim that $\mathcal{A}$ has an automatic presentation.

Exercise 4.3 Show that the following structures are isomorphic to structures that are FOL-definable in the universal structure $(\{0,1\}^*; \preceq, S_0, S_1, \mathsf{EqualLength})$. Give the corresponding FOL-formulae.

(a) $(\mathbb{N}; \leq)$

(b) $(\mathbb{N}; +)$

The next Exercise 4 will not be discussed during the tutorial. Instead, solutions can be submitted via mail to `francesco.kriegel@tu-dresden.de` until 13th June 2016 – individual feedback will be provided.

Exercise 4.4 (Homework) (a) Give a universal structure other than $(\{0,1\}^*; \preceq, S_0, S_1, \mathsf{EqualLength})$.

(b) Show that an automatic structure over the unary alphabet can never be universal.