Term Rewriting Systems

Exercise Sheet 2

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Summer Semester 2016

Exercise 5

Which of the closure operators defined in the lecture commute? Prove or refute the validity of each of the following equations:

a) $(\xrightarrow{+})^{=} = (\xrightarrow{=})^{+}$

b) $(\rightarrow \cup \xrightarrow{-1})^{+} = \xrightarrow{+} \cup (\xrightarrow{+})^{-1}$

c) $(\xrightarrow{-1})^{+} = (\xrightarrow{+})^{-1}$

Exercise 6

Consider the following reduction relations:

- Let M be a set and 2^M the power set of M. We define the reduction relation $\rightarrow_M$ on 2^M as follows: $A \rightarrow_M B$ iff $B \subsetneq A$.
- We define the reduction relation $\rightarrow_{p,q}$ on the non-negative integers as follows:
  $n \rightarrow_{p,q} n - p$ iff $n > p$, and
  $n \rightarrow_{p,q} n - q$ iff $n > q$.

For each of the following properties, describe those sets M and those non-negative integers p, q such that $\rightarrow_M$ and $\rightarrow_{p,q}$ satisfy the following property:

a) terminating,

b) Church-Rosser,

c) normalising, and

d) confluent.

Exercise 7

In the lecture, we defined the set $A := \mathbb{N} \setminus \{0, 1\}$ and the following 'divisibility' relation on A:

$$\rightarrow \; := \{(m, n) \mid m > n \text{ and there is some } \ell \in A \text{ with } n \cdot \ell = m\}.$$

Prove that $\xleftrightarrow{*} = A \times A$.

Exercise 8

Disprove the following claim:

If $(M, \to)$ is a reduction system such that $x \to y$ is decidable, then the set $\{x \in M \mid x \text{ is reducible}\}$ is also decidable.

Hint:

Define a reduction relation $\to$ on the non-negative integers such that $n \to m$ is decidable, but the problem whether n is reducible is undecidable.

Exercise 9

Consider the reduction system $\to_S$ from the lecture with $S := \{abb \to aa, a \to b\}$.
(Note: $\to_S \; = \{(uabbv, uaav) \mid u, v \in \{a, b\}^*\} \cup \{(uav, ubv) \mid u, v \in \{a, b\}^*\}$)

a) Decide whether:

- $ababb \xrightarrow{*}_S bbb$
- $aabb \overset{*}{\leftrightarrow}_S aaaaa$
- $abababbbab \overset{*}{\leftrightarrow}_S aaaaaaaaabababababbababababbababab babababaaaaaaaaaabb$

b) Prove that $\to_S$ is finitely branching.

c) Prove that $\to_S$ is terminating by defining a monotonic mapping $\varphi\colon \{a, b\}^* \to \mathbb{N}$, i.e. a mapping such that $u \to_S v$ implies $\varphi(u) > \varphi(v)$.