# Term Rewriting Systems

## Exercise Sheet 8

Prof. Dr.-Ing. Franz Baader / Dr. rer. nat. Marcel Lippmann
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### Exercise 34

The following problem is known as *Hilbert's 10th problem*, and it has been proved to be undecidable.

> **Given:** One polynomial $P \in \mathbb{Z}[X_1, \dots, X_n]$.
> **Question:** Are there $a_1, \dots, a_n \in \mathbb{N}$ such that $P(a_1, \dots, a_n) = 0$?

Show that the undecidability of Hilbert's 10th problem implies the undecidability of the following problem:

> **Given:** Two polynomials $P, Q \in \mathbb{N}[X_1, \dots, X_n]$ and a decidable set $A \subseteq \mathbb{N} \setminus \{0\}$.
> **Question:** Is $P >_A Q$? That is, is $P(a_1, \dots, a_n) > Q(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in A$?

### Exercise 35

Let $P \in \mathbb{Z}[X]$ be a polynomial with one indeterminate and coefficients in $\mathbb{Z}$.

a) Prove for all $r \in \mathbb{Z}$: if $P(r) = 0$, then $r \mid a_0$, i.e. any root of $P$ divides $a_0$.

b) Devise a decision procedure, which for each polynomial $P \in \mathbb{Z}[X]$ decides whether $P$ has a root in $\mathbb{Z}$.

c) Show that for polynomials with more than one indeterminate, the roots need not satisfy such a property.

### Exercise 36

Consider the TRS $R = \{g(x, g(y, z)) \to g(g(x, y), z),\ g(g(x, y), z) \to g(y, y)\}$. Use a polynomial interpretation to prove that $R$ terminates.

### Exercise 37

A TRS $R$ is called *right-irreducible* if each $r$ with $\ell \to r \in R$ is irreducible. Prove or refute the following claim: If $R$ is right-ground and right-irreducible, then $R$ is terminating.

### Exercise 38

Prove Lemma 5.20 of the lecture:

> Let $\mathcal{A}$ be a monotone polynomial interpretation of $\Sigma$. Then $f^{\mathcal{A}}$ is a monotone function for each $f \in \Sigma$.

**Exercise 39**

Let $R$ be a finite TRS. We use $|t|_f$ to denote the number of occurrences of the symbol $f \in \Sigma$ in the term $t$. Prove the following claim:

> There is a positive integer $k_R$ such that $s \to_R t$ implies $|t|_f \leq k_R(|s|_f + 1)$ for all terms $s$, $t$.

**Exercise 40**

Let $R_{\text{Ack}}$ be the TRS with

$$R_{\text{Ack}} = \{a(0, y) \to s(y), a(s(x), 0) \to a(x, s(0)), a(s(x), s(y)) \to a(x, a(s(x), y))\}.$$

Prove that there is no primitive recursive function that provides an upper bound for the length of reductions given the length of the first term. More precisely, prove that there is no primitive recursive function $f$ such that $t_1 \to_{R_{\text{Ack}}} t_2 \to_{R_{\text{Ack}}} \cdots \to_{R_{\text{Ack}}} t_k$ implies $k \leq f(|t_1|)$.

**Hint:**
Use the previous exercise and the fact the the Ackermann function grows faster than any primitive recursive function.