Term Rewriting Systems

Exercise Sheet 9
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Exercise 41
For each of the following pairs \((s, t)\) of terms, check whether \(s \succeq_{\text{emb}} t\).

a) \(f(x) \succeq_{\text{emb}} a\)

b) \(f(b) \succeq_{\text{emb}} a\)

c) \(g(g(x, y), g(a, f(z))) \succeq_{\text{emb}} g(y, g(a, z))\)

Exercise 42
Prove the following claims:

a) The reduction relation \(\rightarrow_{R_{\text{emb}}}\) given by the TRS

\[
R_{\text{emb}} := \{ f(x_1, \ldots, x_n) \rightarrow x_i \mid n \geq 1, f \in \Sigma^{(n)} \text{ and } 1 \leq i \leq n \}
\]

and the homeomorphic embedding \(\succeq_{\text{emb}}\) are identical, i.e. \(s \rightarrow_{R_{\text{emb}}} t \iff s \succeq_{\text{emb}} t\).

b) \(\succeq_{\text{emb}}\) is a partial order.

c) \(\succeq_{\text{emb}}\) is well-founded. (Prove this without using Kruskal’s Theorem.)

Exercise 43
In the proof of Theorem 5.32, we have used that \(\succeq_{\text{emb}}\) is a well-partial-order. Explain why \(\succeq_{\text{emb}}\) being a well-founded partial order would not have been sufficient.

Exercise 44
In lecture, it was shown that the termination of the TRS \(R := \{ f(f(x)) \rightarrow f(g(f(x))) \}\) cannot be proved using a simplification order.

a) Prove termination using the interpretation method.

b) Is there a polynomial order that can be used to prove termination of \(R\)?
Exercise 45
Prove that polynomial orders are simplification orders if the following properties are satisfied.

- The underlying signature $\Sigma$ contains only function symbols of arity at least 2.
- The domain $A$ does not contain 1, i.e. $A \subseteq \mathbb{N} \setminus \{0, 1\}$.

Are those conditions necessary?

Exercise 46
Prove the first part of Theorem 5.38 of the lecture:

Let $\Sigma$ be a finite signature, $s, t \in T(\Sigma, V)$, and $>_{lpo}$ be a lexicographic path order. We can decide whether $s >_{lpo} t$ in time polynomial in $|s|$ and $|t|$.

Hint: First, show that the condition $s >_{lpo} t$ for all $j$ with $1 \leq j \leq n$ in (LPO2c) can be replaced with $s >_{lpo} t$ for all $j$ with $i \leq j \leq n$ for $i$ such that $s_1 = t_1 \ldots s_{i-1} = t_{i-1}$, and $s_i >_{lpo} t_i$. Use this modified condition to prove that the question whether $s >_{lpo} t$ holds can be decided in time $O(|s| \cdot |t|)$. 

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