

## 9. Exercises for the Course „Logic-based Knowledge Representation“

### Exercise 27:

We define a slight variant of the domino problem: A solution of a domino system  $\mathcal{D} = (D, H, V)$  (as defined in the lecture) *w.r.t. an initial condition*  $a = a_0, \dots, a_n$  is a mapping  $\tau : \mathbb{N} \times \mathbb{N} \rightarrow D$  such that

- $\tau(x, y) = d$  and  $\tau(x + 1, y) = d'$  implies  $(d, d') \in H$ ,
- $\tau(x, y) = d$  and  $\tau(x, y + 1) = d'$  implies  $(d, d') \in V$ , and
- $\tau(i, 0) = a_i$  for  $i \leq n$ .

Prove that it is undecidable whether a domino system  $\mathcal{D}$  has a solution w.r.t. an initial condition  $a$ .

Hints:

- Reduce the halting problem of Turing machines (TMs) to this problem. Consider only TMs which use one-side infinite tapes and are started on the empty tape (for this class of TMs, the halting problem is already undecidable).
- Use the first row  $\{(i, 0) \mid i \in \mathbb{N}\}$  of the  $\mathbb{N} \times \mathbb{N}$  structure to represent the initial tape content, the second row  $\{(i, 1) \mid i \in \mathbb{N}\}$  to represent the tape content after the first step of the TM, and so forth.
- Define additional tile types that, apart from determining the content of a tape cell, represent the position of the TM's head and its current state.
- Use the initial condition to ensure that the TM starts in the initial state and that the tape is initially empty.
- Be careful: the head's position and state must be unique for each step of the computation.

### Exercise 28:

Let  $\mathcal{D} = (D, H, V)$  be a domino system. A solution  $\tau : \mathbb{N} \times \mathbb{N} \rightarrow D$  is called *periodic* iff there exist natural numbers  $h, v > 0$  such that  $\tau(x, y) = \tau(x + h, y) = \tau(x, y + v)$ . Show that

- It is partially decidable whether a domino system has *no* solution.
- It is partially decidable whether a domino system has a *periodic* solution.
- There exist solvable domino systems that admit only non-periodic solutions.

### Exercise 29:

Use a reduction of the domino problem to prove that satisfiability of first-order logic formulas is undecidable. In the reduction, you are allowed to use equality, but no constants and function symbols.

**Exercise 30:**

For the following pairs of  $\mathcal{FL}_0$ -concepts  $C, D$ , use the algorithm from the lecture to decide whether  $C$  is subsumed by  $D$ :

(a)  $C = \forall R.\forall R.(A \sqcap B)$ ,  $D = \forall R.\forall R.A$

(b)  $C = \forall R.A \sqcap \forall S.B$ ,  $D = \forall R.(A \sqcap B) \sqcap \forall S.(A \sqcap B)$

(c)  $C = \forall R.(A \sqcap \forall S.B)$ ,  $D = \forall R.(\forall S.B) \sqcap \forall R.A$

(d)  $C = \forall R.\forall S.A \sqcap \forall S.\forall R.A$ ,  $D = \forall R.\forall R.A \sqcap \forall S.\forall S.A$

(e)  $C = A \sqcap \forall R.A \sqcap \forall S.B$ ,  $D = A \sqcap \forall R.A$