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12. Exercises for the Course „Complexity and Logic“

Exercise 50:

The set of *generalized quantified Boolean formulas (gQBFs)* is the smallest set such that

- every propositional variable is a gQBF;
- if φ and ψ are gQBF and p is a propositional variable, then the following are also gQBFs: $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists p.\varphi$, $\forall p.\varphi$

Observe that, in contrast to the non-generalized QBFs introduced in the lecture, quantifiers can occur anywhere in the formula, and not only in a prefix.

The semantics of gQBFs is defined inductively in terms of truth assignments t as expected: t satisfies

- a propositional variable p if $t(p) = 1$;
- $\neg\varphi$ if t does not satisfy φ;
- $\varphi \wedge \psi$ if t satisfies φ and ψ;
- $\varphi \vee \psi$ if t satisfies φ or ψ;
- $\exists p.\varphi$ if $t[p/0]$ satisfies φ or $t[p/1]$ satisfies φ;
- $\forall p.\varphi$ if $t[p/0]$ satisfies φ and $t[p/1]$ satisfies φ,

where $t[p/b]$ denotes the truth assignment t' defined by setting $t'(p) = b$ and $t'(q) = t(q)$ for all $q \neq p$. A gQBF is *valid* if it is satisfied by all truth assignments. It is a *sentence* if all propositional variables in it are bound by a quantifier.

Show that every gQBF φ which is a sentence can be converted into a non-generalized QBF φ' (as introduced in the lecture) in polynomial time such that φ is valid iff φ' is valid.

Exercise 51:

Prove that every $\mathcal{ALC}$ concept can be converted into an equivalent $\mathcal{ALC}$ concept that uses only the operators $\neg$, $\sqcap$, and $\exists$.

Exercise 52:

For $n > 0$, let the $\mathcal{ALC}$ concept C_n be defined as

$$\begin{aligned} C_n \quad := \quad & \bigsqcap_{1 \leq i \leq n} \forall r^{n-1}.(\exists r.A_i \sqcap \exists r.\neg A_i) \\ \sqcap \; & \bigsqcap_{1 \leq i < n} \bigsqcap_{i \leq j < n} \forall r^j.\big((A_i \rightarrow \forall r.A_i) \sqcap (\neg A_i \rightarrow \forall r.\neg A_i)\big) \end{aligned}$$

where $\forall r^n C$ denotes the n-fold nesting $\forall r.\forall r.\cdots.\forall r.C$, and $C \rightarrow D$ abbreviates $\neg C \sqcup D$. Prove that, for $n > 0$, every model of C_n has at least 2^n elements.

Exercise 53:

Let $\mathcal{ALC}^f$ be the variant of $\mathcal{ALC}$ with all roles interpreted as partial functions (in other words, $\mathcal{ALC}^f$ is the fragment of $\mathcal{ALCF}$ that does not admit the concept constructor $u_1 \doteq u_2$). The *path set* $p(C)$ of an $\mathcal{ALC}^f$ concept C is a set of finite words over the alphabet of role names. It is defined by structural induction as follows:

- $p(A) = \varepsilon$ for all concept names A;
- $p(\neg C) = p(C)$;
- $p(C \sqcap D) = p(C \sqcup D) = p(C) \cup p(D)$;
- $p(\exists r.C) = p(\forall r.C) = \{\varepsilon\} \cup \{r \cdot w \mid w \in p(C)\}$.

Show the following:

(a) every satisfiable $\mathcal{ALC}^f$-concept C has a model $\mathcal{I}$ such that $|\Delta^{\mathcal{I}}| \leq p(C)$;

(b) for every $\mathcal{ALC}^f$-concept C, $|p(C)|$ is bounded by the length of C.

Conclude that every satisfiable $\mathcal{ALC}^f$-concept C has a model whose number of elements is bounded by the length of C.

Exercise 54:

Show that satisfiability of $\mathcal{ALC}^f$-concepts is NP-complete.