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1. Exercises for the lecture **Term Rewriting Systems**

Exercise 1:

For $M = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1, C_2, C_3, C_4, D, E\}$ consider the following relation $\to \subseteq M \times M$:

$$\begin{array}{l} A_i \to B_j \text{ for } i, j \in \{1,2\} \\ A_i \to B_3 \text{ for } i \in \{3,4\} \\ B_1 \to C_1,\ B_2 \to C_2,\ B_2 \to C_3,\ D \to C_4 \\ B_3 \to C_i \text{ for } i \in \{1,2,3,4\} \\ C_i \to E \text{ for } i \in \{3,4\} \end{array}$$

(a) Which of the following properties are satisfied by $\to$?

  (i) finite
  (ii) symmetric
  (iii) antisymmetric
  (iv) reflexive
  (v) irreflexive
  (vi) transitive

(b) A relation r is *smaller than* r' if $r \subseteq r'$. In this case, we also say that r' *contains* r. For p a property on relations and r a relation, the *p closure of* r is the smallest relation containing r that satisfies p.

Describe the following closures of $\to$:

  (i) the symmetric closure of $\to$
  (ii) the reflexive closure of $\to$
  (iii) the transitive closure of $\to$
  (iv) the reflexive-transitive closure of $\to$

(c) Characterise the properties p for which, for any relation r, the p closure of r exists.

(d) For $r, r' \subseteq M \times M$ relations, its *composition* $r \circ r'$ is defined as follows:

$$r \circ r' := \{(\ell, n) \mid \text{there is } m \in M \text{ with } (\ell, m) \in r \text{ and } (m, n) \in r'\}$$

Moreover, we set

$$\begin{array}{rcl} r^0 & := & \{(m,m) \mid m \in M\} \\ r^1 & := & r \\ r^{i+1} & := & r \circ r^i \end{array}$$

Describe $\to^0$, $\to^1$, $\to^2$, and $\to^3$.

Exercise 2:

A *partial order* on M is a reflexive, transitive, and antisymmetric relation on M.
A *linear order* r on M is a partial order that is total, i.e., for any $m, n \in m$, either $(m, n) \in r$ or $(n, m) \in r$.
An *equivalence relation* is a reflexive, transitive, and symmetric relation.
For each of the following relations, say whether they are a partial or linear order or an equivalence relation. For the equivalence relations, describe the equivalence classes.

(a) "is divisible by" on the non-negative integers without 0

(b) "is less than or equal" on the non-negative integers

(c) "occurs as a sub-string in" on words

(d) "is a sibling of" on human beings

(e) "is in the same course as" on students

Exercise 3:

Let $\longrightarrow$ be the *symbolic differentiation relation* introduced in the lecture, i.e.,

$$\mathbf{D}_X(X) \longrightarrow 1 \tag{D1}$$

$$\mathbf{D}_X(Y) \longrightarrow 0 \tag{D2}$$

$$\mathbf{D}_X(\alpha + \beta) \longrightarrow \mathbf{D}_X(\alpha) + \mathbf{D}_X(\beta) \tag{D3}$$

$$\mathbf{D}_X(\alpha \cdot \beta) \longrightarrow \alpha \cdot \mathbf{D}_X(\beta) + \mathbf{D}_X(\alpha) \cdot \beta \tag{D4}$$

(a) Compute the *normal forms* of the following terms:

   (i) $\mathbf{D}_X(((X \cdot X) \cdot X) + (X \cdot X))$

   (ii) $\mathbf{D}_X((X \cdot Y) + (Y \cdot Y))$.

(b) Prove that $\longrightarrow$ is *terminating*.

Exercise 4:

A **group** can be defined by the following identities:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{I1}$$

$$e \circ x \approx x \tag{I2}$$

$$i(x) \circ x \approx e \tag{I3}$$

(a) Prove that groups satisfy

$$x \circ e \approx x \tag{$*$}$$

by showing that $x \circ e$ can be transformed to x using the identities (I1), (I2), and (I3).

(b) Consider the following identity (I3′):

$$x \circ i(x) \approx e \tag{I3'}$$

Prove that (I1), (I2) and (I3′) does not imply $*$.

Hint: Give a counterexample, i.e., a model of (I1), (I2) and (I3'), in which ($*$) does not hold—such a model exists with only two elements.