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2. Exercises for the lecture **Term Rewriting Systems**

Exercise 5:

Consider the following identities:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{R1}$$

$$(x \circ y) \circ x \approx x \tag{R2}$$

Prove or disprove whether the following identities are implied by (R1) and (R2):

(a) $x \circ x \approx x$?

(b) $(x \circ y) \circ z \approx x \circ z$?

Exercise 6:

Let Σ be a finite alphabet. A **word rewrite system** over Σ is a finite set of pairs of words

$$W = \{(u_1, v_1), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*.$$

For W a word rewrite system, we define $\xrightarrow[W]{} \subseteq \Sigma^* \times \Sigma^*$ as follows: $u \xrightarrow[W]{} v$ iff there are $x, y \in \Sigma^*$ and $(u_i, v_i) \in W$ such that $u = xu_iy$ and $v = xv_iy$. Finally, $\leftrightarrow_W$ is the symmetric closure of $\xrightarrow[W]{}$, $\leftrightarrow_W^*$ the reflexive-transitive closure of $\leftrightarrow_W$, and $\xrightarrow[W]{*}$ the transitive-reflexive closure of $\xrightarrow[W]{}$.
Now consider the word rewrite system $W = \{(ab, bb), (ab, a), (b, aba)\}$ over $\{a, b\}$. Decide whether

(a) $b \xrightarrow[W]{*} aaaa$,

(b) $abb \xrightarrow[W]{*} bbba$,

(c) $bbbb \xrightarrow[W]{*} aa$, and

(d) $bbbb \leftrightarrow_W^* aa$.

Exercise 7:

Consider the following identities E:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{W1}$$

$$(x \circ y) \circ x \approx x \circ y \tag{W2}$$

$$x \circ x \approx x \tag{W3}$$

Devise a decision procedure for the following problem: given two terms s and t over $\circ$, can they be rewritten to the same term using (W1), (W2), and (W3)?
Hint: (W1) implies that parentheses are obsolete, and we can thus work on words/strings rather than on terms. Then we can try to find a way to get short words by applying the identities (W2) and (W3). Using this knowledge, we can describe a procedure that compare such “shortest forms” of two terms.

Exercise 8:

Which of the closure operator defined in the lecture commute? Prove or disprove the validity of each of the following equations:

(a) $(\xrightarrow{+})^{=} = (\xrightarrow{=})^{+}$

(b) $(\longrightarrow \cup \xrightarrow{-1})^{+} = (\xrightarrow{+}) \cup (\xrightarrow{+})^{-1}$

(c) $(\xrightarrow{-1})^{+} = (\xrightarrow{+})^{-1}$

Exercise 9:

Consider the following reduction relations:

- Let M be a set and 2^M the set of all subsets of M. We define the reduction relation $\underset{M}{\longrightarrow}$ on 2^M as follows: $A \underset{M}{\longrightarrow} B$ iff $B \subsetneq A$.
- We define the reduction relation $\underset{p,q}{\longrightarrow}$ on the non-negative integers as follows:

  $n \underset{p,q}{\longrightarrow} n-p \quad$ iff $\quad n > p$, and
  $n \underset{p,q}{\longrightarrow} n-q \quad$ iff $\quad n > q$.

For each of the following properties, describe those sets M and those non-negative integers p, q such that $\underset{M}{\longrightarrow}$ and $\underset{p,q}{\longrightarrow}$ satisfy this property.

(a) terminating?

(b) Church-Rosser?

(c) normalising?

(d) confluent?