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3. Exercises for the lecture **Term Rewriting Systems**

Exercise 10:

In the lecture, we set $A := \mathbb{N} \setminus \{0, 1\}$ and considered the following "divisibility" reduction relation on A:

$$\longrightarrow := \{(m, n) \in A \times A \mid m > n \text{ and there is some } \ell \in A \text{ with } n \cdot \ell = m\}.$$

Prove that $\overset{*}{\longleftrightarrow} = A \times A$.

Exercise 11:

Disprove the following claim:

"If $(M, \longrightarrow)$ is a reduction system such that $x \longrightarrow y$ is decidable, then the set $\{x \in M \mid x \text{ is reducible}\}$ is also decidable."

Hint: define a reduction relation $\longrightarrow$ on the non-negative integers such

- $n \longrightarrow m$? is decidable, but
- is n reducible? is undecidable

Exercise 12:

Consider the Semi-Thue system $\underset{S}{\longrightarrow}$ from the lecture with $S := \{abb \longrightarrow aa, a \longrightarrow b\}$ (remember: $\underset{S}{\longrightarrow} = \{(uabbv, uaav) \mid u, v \in \{a, b\}^*\} \cup \{(uav, ubv) \mid u, v \in \{a, b\}^*\}$)

(a) Prove that $\underset{S}{\longrightarrow}$ is finitely branching.

(b) Prove that $\underset{S}{\longrightarrow}$ is terminating by defining a monotone mapping $\varphi : \{a, b\}^* \longrightarrow \mathbb{N}$, i.e., a mapping such that $u \underset{S}{\longrightarrow} v$ implies $\varphi(u) > \varphi(v)$.

Exercise 13:

Let $\mathbf{ack} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Ackermann function, i.e.,

$$\mathbf{ack}(n, m) = \begin{cases} m + 1 & \text{, if } n = 0 \\ \mathbf{ack}(n-1, 1) & \text{, if } m = 0 \\ \mathbf{ack}(n-1, \mathbf{ack}(n, m-1)) & \text{, otherwise} \end{cases} .$$

Choose an appropriate order on pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ and use well-founded induction to prove that the function **ack** is well-defined (Hint: the Ackermann function is well-defined if for each pair of non-negative integers it determins a unique value).

Exercise 14:

Prove Lemma 2.21 from the lecture:
Lemma 2.21: If $>_A$ is a strict order on A and $>_B$ is a strict order on B, then the lexicographic product $>_{A \times B}$ is a strict order on $A \times B$.