11. Exercises for the Course
‘Automata and Logic’

Exercise 52:
Let \( L := (a+b)^\omega \cup (b+a)^\omega \). Use the proof of Theorem 5.4 to construct a closed S1S-formula \( \varphi \) with \( L_\omega(\varphi) = L \).

Exercise 53:
A Rabin-automaton is a tuple \( A := (Q, \Sigma, I, \Delta, \Omega) \), where \( Q, \Sigma, I, \) and \( \Delta \) are defined as for non-deterministic Büchi automata and \( \Omega := \{(F_1, G_1), \ldots, (F_n, G_n)\} \) is a finite set of pairs \((F_i, G_i)\) such that \( F_i, G_i \subseteq Q \). For a word \( \alpha \), \( \text{path}_A(\alpha) \) denotes the set of all paths in \( A \) labelled with \( \alpha \). For a path \( p \in \text{path}_A(\alpha) \), let \( \text{inf}(p) \) denote the set of all states that are visited infinitely often. The \( \omega \)-language \( L_\omega(A) \) recognized by a Rabin automaton is defined as
\[
\{ \alpha \in \Sigma^\omega \mid \exists i \in \{1, \ldots, n\} \exists p \in \text{path}_A(\alpha) : \text{inf}(p) \cap F_i \neq \emptyset \text{ and } \text{inf}(p) \cap G_i = \emptyset \}.
\]
Show that every language recognized by a Rabin-automaton is also recognized by a Büchi-automaton by constructing, for a given Rabin-automaton \( A \), an S1S-formula \( \varphi_A \) defining the language \( L_\omega(A) \).

Exercise 54:
Let \( \Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \) be an alphabet with arity function, where \( \Sigma_0 = \{x, y, z\}, \Sigma_1 = \{\neg\} \), and \( \Sigma_2 = \{\land, \lor\} \). Define tree automata (either LR or RL) recognizing the tree-languages consisting of the following trees:
(a) trees containing the symbol \( \lor \) exactly once;
(b) trees containing the symbol \( \neg \) at least once on every path of the tree;
(c) trees describing satisfiable propositional formulas.

Exercise 55:
Let \( A = (Q, \Sigma, I, \Delta, F) \) be the non-deterministic LR-tree automaton given by:
- \( Q = \{0, 1\} \),
- \( \Sigma = \{f, x, y\} \) with \( \nu(f) = 2 \) and \( \nu(x) = \nu(y) = 0 \),
- \( I(x) = \{0, 1\} \) and \( I(y) = \{0\} \),
- \( \Delta_f(0, 0) = \{0\}, \Delta_f(0, 1) = \{1, 0\}, \Delta_f(1, 0) = \{1, 0\}, \) and \( \Delta_f(1, 1) = \{1\} \),
- \( F = \{1\} \).
Do the following:
(a) Adapt the standard powerset construction from finite automata on words to LR-tree automata and use it to construct a deterministic LR-tree automaton \( A' \) such that \( L(A) = L(A') \).

(b) Try to apply a similar construction to the RL-automaton from Example 6.10. Explain why this method fails for RL-automata.

**Exercise 56:**
Let \( \Sigma \) be an alphabet with arity function, \( x, y \in \Sigma_0 \) and \( U, V, W \subseteq T_\Sigma \). Prove or refute:

- \( U \cdot^x (V \cup W) = (U \cdot^x V) \cup (U \cdot^x W) \),
- \( (U \cdot^x V) \cdot^y W = U \cdot^x (V \cdot^y W) \),
- \( (U^\ast \cdot^x)^\ast y = (U^\ast y)^\ast \cdot^x \).