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3. Exercises for the Course 'Automata and Logic'

Exercise 11:

Let Σ be an alphabet and $(M, \circ, 1)$ a monoid. Prove that every function $f : \Sigma \to M$ can be uniquely extended to a (monoid-)homomorphism $\Phi : \Sigma^* \to M$.

Exercise 12:

Let $\Sigma := \{a, b\}, M := \{0, 1, 2\}$, and define $\circ : M \times M \to M$ as $x \circ y := (x + y) \mod 3$. We define mappings $\Phi, \Phi' : \Sigma^* \to M$ by setting $\Phi(w) := |w| \mod 3$ and $\Phi'(w) := |w|_a \mod 3$, where $|w|$ denotes the *length* of w and $|w|_a$ the number of times the symbol "a'' appears in w.

(a) Show that both Φ and Φ' are monoid homomorphisms from $(\Sigma^*, \cdot, \varepsilon)$ into $(M, \circ, 0)$.

(b) For each of the languages $\Phi^{-1}(\{0, 2\}), \Phi^{-1}(\{1\})$ and $(\Phi')^{-1}(\{1\})$ devise a finite automaton that recognizes the language.

Exercise 13:

For a language $L \subseteq \Sigma^*$, we use $\bar{L}$ to denote the complement of L: $\bar{L} := \Sigma^* \setminus L$. Let Σ be an alphabet, $L \subseteq \Sigma^*$ a language and $(M, \circ, 1)$ a monoid. Prove that if L is accepted by $(M, \circ, 1)$, then $\bar{L}$ is also accepted by $(M, \circ, 1)$.

Exercise 14:

Determine the syntactic monoid of the language a^*ba^*.

Exercise 15:

Let $L \subseteq \Sigma^*$ and $\approx$ be an equivalence relation on Σ^*. Consider the following property:

$$\text{for all } u, v \in \Sigma^*, \text{ if } u \in L \text{ and } u \approx v, \text{ then } v \in L. \qquad (*)$$

(a) The proof of Corollary 1.13 from the lecture depends on the fact that the syntactical congruence $\sim_L$ has Property $(*)$. Prove this.

(b) Show that $\sim_L$ is the coarsest congruence relation with Property $(*)$.

(c) Show that the Nerode right congruence ρ_L is the coarsest right congruence with Property $(*)$.

Note: An equivalence relation $\approx_2$ is *coarser* than $\approx_1$ if, for every x, y, $x \approx_1 y$ implies $x \approx_2 y$ (in particular, $\approx_2$ has at most as many equivalence classes as $\approx_1$).

Exercise 16:

Show that any submonoid of a finite group is also a group.