3. Exercises for the Course
‘Automata and Logic’

Exercise 11:
Let \( \Sigma \) be an alphabet and \((M, \odot, 1)\) a monoid. Prove that every function \( f : \Sigma \to M \) can be uniquely extended to a (monoid-)homomorphism \( \Phi : \Sigma^* \to M \).

Exercise 12:
Let \( \Sigma := \{a, b\} \), \( M := \{0, 1, 2\} \), and define \( \odot : M \times M \to M \) as \( x \odot y := (x + y) \mod 3 \). We define mappings \( \Phi, \Phi' : \Sigma^* \to M \) by setting \( \Phi(w) := \vert w \vert \mod 3 \) and \( \Phi'(w) := \vert w \vert_a \mod 3 \), where \( \vert w \vert \) denotes the length of \( w \) and \( \vert w \vert_a \) the number of times the symbol “a” appears in \( w \).

(a) Show that both \( \Phi \) and \( \Phi' \) are monoid homomorphisms from \((\Sigma^*, \cdot, \varepsilon)\) into \((M, \odot, 0)\).

(b) For each of the languages \( \Phi^{-1}(\{0, 2\}), \Phi'^{-1}(\{1\}) \) and \( (\Phi')^{-1}(\{1\}) \) devise a finite automaton that recognizes the language.

Exercise 13:
For a language \( L \subseteq \Sigma^* \), we use \( \bar{L} \) to denote the complement of \( L \): \( \bar{L} := \Sigma^* \setminus L \). Let \( \Sigma \) be an alphabet, \( L \subseteq \Sigma^* \) a language and \((M, \odot, 1)\) a monoid. Prove that if \( L \) is accepted by \((M, \odot, 1)\), then \( \bar{L} \) is also accepted by \((M, \odot, 1)\).

Exercise 14:
Determine the syntactic monoid of the language \( a^*ba^* \).

Exercise 15:
Let \( L \subseteq \Sigma^* \) and \( \approx \) be an equivalence relation on \( \Sigma^* \). Consider the following property:

\[
\text{for all } u, v \in \Sigma^*, \text{ if } u \in L \text{ and } u \approx v, \text{ then } v \in L. \tag{\ast}
\]

(a) The proof of Corollary 1.13 from the lecture depends on the fact that the syntactical congruence \( \sim_L \) has Property (\ast). Prove this.

(b) Show that \( \sim_L \) is the coarsest congruence relation with Property (\ast).

(c) Show that the Nerode right congruence \( \rho_L \) is the coarsest right congruence with Property (\ast).

Note: An equivalence relation \( \approx_2 \) is coarser than \( \approx_1 \) if, for every \( x, y \), \( x \approx_1 y \) implies \( x \approx_2 y \) (in particular, \( \approx_2 \) has at most as many equivalence classes as \( \approx_1 \)).

Exercise 16:
Show that any submonoid of a finite group is also a group.