4. Exercises for the Course
‘Automata and Logic’

Exercise 17:
Let $V$ be an M-variety. Show that $L(V)_\Sigma$ is closed under union without using Theorem 1.22 from the lecture.

Exercise 18:
Let $\Sigma$ be an alphabet. Prove or refute the following claims:

- Every regular language $L \subseteq \Sigma^*$ is accepted by its syntactic monoid.
- If $L \subseteq \Sigma^*$ is accepted by a finite group, then the syntactic monoid of $L$ is a finite group.
- For every regular language $L \subseteq \Sigma^*$, the syntactic monoid $M_L$ is the smallest monoid accepting $L$; i.e. for every monoid $M$ accepting $L$, we have $|M_L| \leq |M|$.
- Let $\overline{w}$ denote the mirror image of the word $w$; that is, for $w = a_1 \ldots a_n$, $\overline{w} = a_n \ldots a_1$. For a language $L \subseteq \Sigma^*$, we define $\overline{L} := \{ \overline{w} \mid w \in L \}$. **Claim:** if the minimal automaton for $L$ has $n$ states, then the minimal automaton for $\overline{L}$ has also $n$ states.

Exercise 19:
Let $V$ be the M-variety of all commutative finite groups. Show that there exist a language $L \subseteq \{a\}^*$ such that $L \in L(V)_{\{a\}}$ but $L \notin L(V)_{\{a,b\}}$.

Exercise 20:
Prove or disprove the following: there is a language $L \subseteq \{a, b\}^*$ such that its syntactic semigroup $S_L$ and its syntactic monoid $M_L$ are isomorphic.

Exercise 21:
For each of the following words over the alphabet $\{0, 1\}^k$, give a corresponding interpretation over the predicate symbols $P_1, \ldots, P_k$ as discussed in the lecture:

- $k = 2$: $(1, 1), (1, 1), (0, 1), (1, 0)$
- $k = 3$: $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$
- $k = 3$: $(1, 1, 0), (1, 0, 1), (1, 1, 1), (1, 1, 0)$

Describe all interpretations that correspond to words of the language $L(((0, 1) \cdot (1, 0))^+) \subseteq (\{0, 1\}^2)^+$.
Exercise 22:

Let $\Sigma = \{a, b\}$. For each of the following regular expressions $r_i$, give a first-order formula $\varphi_i$ such that $L(r_i) = L(\varphi_i)$.

(a) $r_1 = \Sigma^*$,
(b) $r_2 = \varepsilon$,
(c) $r_3 = (abb^*)^*$,
(d) $r_4 = a^*b^* + b^*a^*$,
(e) $r_5 = (aaa \cdot \Sigma^*) + b^*$. 