Fuzzy Description Logics

Lecture Notes, TU Dresden

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Contents

1 Description Logics

Description Logics (DLs) are a family of knowledge representation formalisms that have formal and well-understood semantics. There are several members of this family (that is, several description logics), with different expressiveness and complexity properties. These are distinguished by the type of constructors that they allow.

In DL, knowledge is represented using *concepts* (which correspond to unary relations) such as Heroe or Strong and *roles* (binary relations) like hasSidekick. Concepts can be combined with the help of *constructors* to obtain more complex expressions. For example, one can express the class of all *heroes that have a strong sidekick* as:

$$\text{Heroe} \sqcap \exists \text{hasSidekick}.\text{Strong}.$$

Axioms are then used to impose restrictions in the concepts, and include some individuals as well. For example, we can state that *Batman is a superheroe*

$$\text{Superheroe}(\text{batman})$$

and that *heroes only have heroic sidekicks*

$$\text{Heroe} \sqsubseteq \forall \text{hasSidekick}.\text{Heroe}.$$

We will now formally introduce the "basic" DL $\mathcal{ALC}$, and its sublogic $\mathcal{EL}$.

1.1 $\mathcal{ALC}$

$\mathcal{ALC}$ is the smallest propositionally closed DL. In it, concepts are built using the constructors *negation* ($\neg$), *conjunction* ($\sqcap$), *disjunction* ($\sqcup$), *existential-* ($\exists$) and *value restrictions* ($\forall$).

Formally, consider the three mutually disjoint sets N_C, N_R and N_I of *concept-*, *role-*, and *individual-names*, respectively. Then, (complex) *concept descriptions* are built from these sets inductively, as follows:

- every concept name $A \in N_C$ is a concept description;

- $\top$ and $\bot$ are concept descriptions

- if C, D are concept descriptions and $r \in N_R$, then

$$\neg C,\ C \sqcap D,\ C \sqcup D,\ \exists r.C,\ \forall r.C$$

  are all concept descriptions.

The semantics of $\mathcal{ALC}$ is based on *interpretations*. Intuitively, an interpretation is a function that assigns to every concept, a set of individuals from a given interpretation domain (that is, those individuals that satisfy the concept).

Formally, an *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain* and $\cdot^{\mathcal{I}}$ is a function that assigns:

- to every concept name $A \in N_C$ a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$,

- to every role name $r \in N_R$ a set of pairs $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and

- to every individual name $a \in N_I$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

This function is extended to concept descriptions as follows:

- $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $\bot^{\mathcal{I}} = \emptyset$,

- $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$,

- $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$,

- $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$,

- $(\exists r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}.(x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$,

- $(\forall r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}.(x,y) \in r^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$,

For example, consider the interpretation described by the following figure.

[Figure hand-drawn]

It then follows that

$$(\forall \mathsf{hasSidekick}.\mathsf{Heroe})^{\mathcal{I}} = \{\mathsf{batman}, \mathsf{robin}, \mathsf{harleyquinn}\}$$

and

$$(\mathsf{Heroe} \sqcap \exists \mathsf{hasSidekick}.\top)^{\mathcal{I}} = \{\mathsf{batman}\}.$$

We consider two kinds of axioms: *assertional axioms* of the form $C(a)$ where C is a concept description and $a \in \mathsf{N_I}$, and *general concept inclusion* axioms (GCIs) of the form $C \sqsubseteq D$, where C, D are two concept descriptions.

An *ABox* is a set of assertional axioms, a *TBox* is a set of GCIs, and an *ontology* is the union of an ABox and a TBox.

We will sometimes use the expression $C \equiv D$ to express the two GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$.

Axioms are used to restrict the set of interpretations that we are interested in. An interpretation $\mathcal{I}$ satisfies the assertional axiom $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$; it satisfies the GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. We say that $\mathcal{I}$ is a model of the ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$.

For example, the interpretation in the previous figure satisfies the GCI

$$\mathsf{Heroe} \sqsubseteq \forall \mathsf{hasSidekick}.\mathsf{Heroe}$$

but not the axiom *only heroes can have sidekicks*

$$\exists \mathsf{hasSidekick}.\top \sqsubseteq \mathsf{Heroe}.$$

The knowledge of a representation domain is stored in an ontology (hence, it is also called a *knowledge base*). Reasoning is then restricted to consider only models of this ontology.

Two of the main reasoning tasks are *ontology consistency* (deciding whether an ontology has a model) and *concept subsumption* (deciding whether a concept is a subconcept of another one in every model of the ontology). In reality for this last reasoning problem, only the TBox part of the ontology is relevant.

Formally, we say that a concept C is *subsumed by* a concept D w.r.t. a TBox $\mathcal{T}$ (in symbols $C \sqsubseteq_{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.

It is a simple exercise to show that subsumption can be reduced to ontology consistency in the following way: $C \sqsubseteq_{\mathcal{T}} D$ iff $\mathcal{T} \cup \{C \sqcap \neg D(a)\}$ is inconsistent.

Complexity-wise, this simple logic is already (relatively) hard: subsumption and ontology consistency are indeed EXPTIME-complete problems. In order to regain tractability, one can consider the sub-boolean DL $\mathcal{EL}$.

1.2 $\mathcal{EL}$

$\mathcal{EL}$ is a sublogic of $\mathcal{ALC}$ that allows only for the constructors $\top$, $\sqcap$ and $\exists$. For example, we can express that only heroes have sidekicks

$$\exists \mathsf{hasSidekick}.\top \sqsubseteq \mathsf{Heroe}$$

but *not* that every sidekick is a hero. In fact, this logic is not capable of expressing negations. Thus, every ontology expressed in $\mathcal{EL}$ is necessarily consistent. For this reason, the main decision problem considered in this setting is concept subsumption.

$\mathcal{EL}$ concept subsumption is decided using a *completion algorithm* that runs in polynomial time. This algorithm assumes that the TBox is in *normal form*; that is, that all the axioms are of the form

$$
\begin{aligned}
A_1 \sqcap A_2 &\sqsubseteq B \\
A_1 &\sqsubseteq \exists r.B \text{ or} \\
\exists r.A_1 &\sqsubseteq B
\end{aligned}
$$

where $A_1, A_2, B \in \mathsf{N_C} \cup \{\top\}$.

Every $\mathcal{EL}$ TBox can be equivalently transformed to normal form using the following normalization rules:

NF1 $C \sqcap \hat{D} \sqsubseteq E \rightsquigarrow \hat{D} \sqsubseteq A, C \sqcap A \sqsubseteq E,$

NF2 $\exists r.\hat{D} \sqsubseteq E \rightsquigarrow \hat{D} \sqsubseteq A, \exists r.A \sqsubseteq E,$

NF3 $B \sqsubseteq \exists r.\hat{C} \rightsquigarrow A \sqsubseteq \hat{C}, B \sqsubseteq \exists r.A,$

NF4 $\hat{C} \sqsubseteq \hat{D} \rightsquigarrow \hat{C} \sqsubseteq A, A \sqsubseteq \hat{D}$,

NF5 $C \sqsubseteq D \sqcap E \rightsquigarrow C \sqsubseteq D, C \sqsubseteq E$,

where $\hat{C}, \hat{D} \notin \mathsf{N_C} \cup \{\top\}$ and A is a new concept name.

Given a TBox in normal form, the algorithm builds a completion graph as follows:

- for every concept $A \in \mathsf{N_C} \cup \{\top\}$, the set $S(A)$ is initialized to $\{A, \top\}$, (subsumers of A)

- for every pair of concepts $A, B \in \mathsf{N_C} \cup \{\top\}$, the set $R(A, B)$ is initialized to $\emptyset$, (roles where B is successor of A),

- these sets are then extended using the following completion rules, until no rule is applicable:

  R1 if $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}$ and $A_1, A_2 \in S(A)$, then add B to $S(A)$,

  R2 if $A_1 \sqsubseteq \exists r.B \in \mathcal{T}$ and $A_1 \in S(A)$, then add r to $R(A, B)$, and

  R3 if $\exists r.A_1 \sqsubseteq B \in \mathcal{T}$, $A_1 \in S(A_2)$, and $r \in R(A, A_2)$, then add B to $S(A)$.

Intuitively, $S(A)$ stores the set of all subsumers w.r.t. $\mathcal{T}$ of A, while $R(A, B)$ stores all the roles r such that $A \sqsubseteq_{\mathcal{T}} \exists r.B$. The idea behind the rules is best explained through the following diagram [hand-drawn diagram].

This algorithm terminates after polynomially many rule applications, and is such that, for every two concept names A, B, $A \sqsubseteq_{\mathcal{T}} B$ iff $B \in S(A)$. Thus, this algorithm decides not only one subsumption relation, but all subsumptions between concept names. In other words, it is capable of *classifying* the whole TBox.

2 Gödel Description Logics

2.1 Motivation

Recall that one of the main goals of DL is to represent knowledge from a domain. A big challenge in knowledge representation is how to deal with imprecise and vague concepts that appear in almost any real-life domain. One such concept is "Strong". It is not possible to give a precise, clear-cut definition of what "Strong" is. However, we still know that, for example, Superman is strong. How about Batman, or Joker?

A solution to this problem is to use intermediate *degrees of membership* to a concept, expressed by numbers in the unit interval $[0, 1]$. The intuition is that the degree expresses how much an individual actually belongs to a concept. Hence, we can say that "Superman is strong with degree 1", while "Joker is strong with degree 0.5".

Obviously, the interpretation of the different concept constructors must also be adapted accordingly.

2.2 Gödel Semantics

The Gödel semantics for DLs are based on interpretations for the concept- and role-names, as with the classical semantics introduced in the previous chapter. The difference is the range of the interpretation function.

For classical DLs, we defined an interpretation as a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\cdot^{\mathcal{I}}$ maps every concept name A to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and every role name r to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. We can view these two sets as their characteristic functions, thus, $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to \{0, 1\}$ and $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \{0, 1\}$, where 0 expresses that the element does not belong to the set, and 1 that it belongs.

We extend this notion to the degrees of membership in the interval $[0, 1]$ in the natural way.

Definition 2.1 (fuzzy semantics). A *fuzzy interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called the *domain* and $\cdot^{\mathcal{I}}$ is a function mapping:

- every concept name $A \in \mathsf{N_C}$ to a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, and

- every role name $r \in \mathsf{N_R}$ to a function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.

$\triangle$

What characterizes the Gödel semantics is how this interpretation function is extended to complex concepts. Consider first the conjunction constructor $\sqcap$. In classical DL, an element belongs to $(C \sqcap D)^{\mathcal{I}}$ iff it belongs to $C^{\mathcal{I}}$ and to $D^{\mathcal{I}}$. Seeing $\mathcal{I}$ as a characteristic function, it follows that

$$(C \sqcap D)^{\mathcal{I}}(x) = \min(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)).$$

Thus, a natural way to interpret conjunction is through the operator min. Likewise, disjunction can be interpreted using max. The interpretation of negation and implication require a more complex intuition that should become clear in the next chapter. The negation is interpreted through the *Gödel negation*

$$\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{otherwise,} \end{cases}$$

and implication through the *residuum*

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

In summary, the interpretation function is extended to concept descriptions as follows. For every $x \in \Delta^{\mathcal{I}}$:

- $\top^{\mathcal{I}}(x) = 1$, $\bot^{\mathcal{I}}(x) = 0$,

- $(\neg C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$,

- $(C \sqcap D)^{\mathcal{I}}(x) = \min(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$,

- $(C \sqcup D)^{\mathcal{I}}(x) = \max(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$,

- $(\exists r.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}} \{\min(r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y))\}$,

- $(\forall r.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}$.

We also extend the notion of membership degree to the axioms in ontologies. In this case, we do not ask axioms to hold always (with a degree 1) but rather give a lower bound of the degree that they must satisfy.

Definition 2.2 (fuzzy ontology). A *(labeled) assertion* is of the form $\langle C(a), q \rangle$ or $\langle r(a,b), q \rangle$, where C is a concept, r a role name, $a, b \in \mathsf{N_I}$, and $q \in [0,1]$.

A *(labeled) GCI* is of the form $\langle C \sqsubseteq D, q \rangle$, where C, D are two concepts and $q \in [0,1]$.

A *fuzzy ABox* is a finite set of assertions, a *fuzzy TBox* is a finite set of GCIs, and a *fuzzy ontology* is the union of an ABox and a TBox. $\triangle$

For example, the axiom $\langle \mathsf{Strong}(\mathsf{batman}), 0.9 \rangle$ expresses that Batman has a "degree of strength" of *at least* 0.9. We can also relate the degrees of strength of different individuals, for instance through the axiom

$$\langle \mathsf{Heroe} \sqcap \mathsf{Strong} \sqsubseteq \forall \mathsf{hasNemesis.Strong}, 0.8 \rangle.$$

According to this axiom, the stronger a hero is, the stronger his enemies must be.

Intuitively, $\langle \alpha, q \rangle$ expresses that the axiom α holds with a degree *at least* q. Formally, an interpretation $\mathcal{I}$ *satisfies* the axiom $\langle C(a), q \rangle$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq q$; it *satisfies* $\langle r(a,b), q \rangle$ if $r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq q$; and it *satisfies* the GCI $\langle C \sqsubseteq D, q \rangle$ if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq q$ for every $x \in \Delta^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$.

2.3 Reasoning in Gödel $\mathcal{EL}$

As in the crisp case, the interesting reasoning task in Gödel $\mathcal{EL}$ is concept subsumption, but in this case, the degree with which the subsumption holds is also relevant. To be precise, given two concepts C, D and $q \in [0,1]$, we say that C is *subsumed to a degree q* by D w.r.t. a TBox $\mathcal{T}$ (denoted as $\langle C \sqsubseteq_{\mathcal{T}} D, q \rangle$) if for every model $\mathcal{I}$ of $\mathcal{T}$ it holds that

$$\inf_{x \in \Delta^{\mathcal{I}}} (C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)) \geq q.$$

Moreover, it is interesting the greatest possible degree to which a subsumption holds; that is a value $q \in [0,1]$ such that $\langle C \sqsubseteq_{\mathcal{T}} D, q \rangle$ and for every $q' > q$ $\langle C \sqsubseteq_{\mathcal{T}} D, q' \rangle$ does not hold. This is called the *best subsumption degree*.

We will show that the completion algorithm sketched in the previous chapter can be adapted to classify the TBox in the sense that the best subsumption degree of every pair of concept names A, B is computed.

The completion algorithm assumes that all axioms are in *normal form*; that is, that they are of the shape:

$$\langle A_1 \sqcap A_2 \ \sqsubseteq \ B, q \rangle$$
$$\langle A_1 \ \sqsubseteq \ \exists r.B, q \rangle \text{ or}$$
$$\langle \exists r.A_1 \ \sqsubseteq \ B, q \rangle$$

where $A_1, A_2, B \in \mathsf{N_C} \cup \{\top\}$, and $q \in [0,1]$.

Consider the normalization rules

NF1 $\left\langle C \sqcap \hat{D} \sqsubseteq E, q \right\rangle \rightsquigarrow \left\langle \hat{D} \sqsubseteq A, q \right\rangle, \langle C \sqcap A \sqsubseteq E, q \rangle,$

NF2 $\left\langle \exists r.\hat{D} \sqsubseteq E, q \right\rangle \rightsquigarrow \left\langle \hat{D} \sqsubseteq A, q \right\rangle, \langle \exists r.A \sqsubseteq E, q \rangle,$

NF3 $\left\langle B \sqsubseteq \exists r.\hat{C}, q \right\rangle \rightsquigarrow \left\langle A \sqsubseteq \hat{C}, q \right\rangle, \langle B \sqsubseteq \exists r.A, q \rangle,$

NF4 $\left\langle \hat{C} \sqsubseteq \hat{D}, q \right\rangle \rightsquigarrow \left\langle \hat{C} \sqsubseteq A, q \right\rangle, \left\langle A \sqsubseteq \hat{D}, q \right\rangle,$

NF5 $\langle C \sqsubseteq D \sqcap E, q \rangle \rightsquigarrow \langle C \sqsubseteq D, q \rangle, \langle C \sqsubseteq E, q \rangle,$

where $\hat{C}, \hat{D} \notin \mathsf{N_C} \cup \{\top\}$, A is a new concept name, and $q \in [0,1]$.

Theorem 2.3. *For any $\mathcal{EL}$ TBox $\mathcal{T}$, the normalization rules produce in polynomial time a TBox $\mathcal{T}'$ in normal form such that, for every $A, B \in \{P \mid P \text{ is a concept name in } \mathcal{T}\} \cup \{\top\}$ and every $q \in [0,1]$ it holds that*

$$\langle A \sqsubseteq_{\mathcal{T}} B, q \rangle \text{ iff } \langle A \sqsubseteq_{\mathcal{T}'} B, q \rangle.$$

Proof. [Exercise!–identical to crisp version] $\qquad \square$

The algorithm builds a generalization of a completion graph, where the elements have also an associated degree. More precisely, we will construct

- for every concept $A \in \mathsf{N_C} \cup \{\top\}$ appearing in the TBox, a subset $S(A)$ of $\mathsf{N_C} \cup \{\top\} \times [0,1]$, and

- for every pair of concepts $A, B \in \mathsf{N_C} \cup \{\top\}$ a subset of $\mathsf{N_R} \cup [0,1]$.

The intuition is again that these sets store the known subsumption relations between concepts in the sense that $(B, q) \in S(A)$ implies $\langle A \sqsubseteq_{\mathcal{T}} B, q \rangle$, and $(r, q) \in R(A, B)$ implies $\langle A \sqsubseteq_{\mathcal{T}} \exists r.B, q \rangle$.

The algorithm initializes these sets by including only the obvious subsumption relations; that is, initially we have

- $S(A) = \{(A, 1), (\top, 1)\}$,

- $R(A, B) = \emptyset$,

for all $A, B \in \mathsf{N_C} \cup \{\top\}$.

The sets are then extended using the following three rules, until no rule is applicable anymore.

R1 if $\langle A_1 \sqcap A_2 \sqsubseteq B, q \rangle \in \mathcal{T}$, $(A_1, q_1), (A_2, q_2) \in S(A)$, $p = \min(q, q_1, q_2)$, and there is no $p' \geq p$ with $(B, p') \in S(A)$, then add (B, p) to $S(A)$,

R2 if $\langle A_1 \sqsubseteq \exists r.B, q \rangle \in \mathcal{T}$, $(A_1, q_1) \in S(A)$, $p = \min(q, q_1)$, and there is no $p' \geq p$ with $(r, p') \in R(A, B)$, then add (r, p) to $R(A, B)$,

R3 if $\langle \exists r.A_1 \sqsubseteq B, q \rangle \in \mathcal{T}$, $(A_1, q_1) \in S(A_2)$, $(r, q_2) \in R(A, A_2)$, $p = \min(q, q_1, q_2)$, and there is no $p' \geq p$ with $(B, p') \in S(A)$, then add B to $S(A)$.

Notice that we can have several pairs $(B, q_1), (B, q_2), \ldots$ in the same completion set. Additionally, the resulting sets depend on the order in which the rules were applied. For example, consider the TBox $\mathcal{T}$ with the axioms

$$\langle A \sqsubseteq B, 1 \rangle \quad \langle B \sqsubseteq C, 0.9 \rangle \quad \langle A \sqsubseteq C, 1 \rangle.$$

If we apply the rules in this order, we will obtain

$$S(A) = \{(A, 1), (\top, 1), (B, 1), (C, 0.9), (C, 1)\}.$$

However, if the third axiom is used first, the resulting set is

$$S(A) = \{(A, 1), (\top, 1), (B, 1), (C, 1)\}.$$

The completion rules maintain the intuition described before, of the completion sets storing the explicit subsumption relations between concepts.

Lemma 2.4. *The completion algorithm preserves the following invariants:*

- $(B, q) \in S(A)$ *implies* $\langle A \sqsubseteq_{\mathcal{T}} B, q \rangle$,

- $(r, q) \in R(A, B)$ *implies* $\langle A \sqsubseteq_{\mathcal{T}} \exists r.B, q \rangle$.

Proof. The initialization obviously satisfies the invariants, hence we need only to show that every rule application preserves them. Consider the third rule.

From the invariants, we know that if $(A_1, q_1) \in S(A_2)$, then it holds that $\langle A_2 \sqsubseteq_{\mathcal{T}} A_1, q_1 \rangle$. We show that this implies that

$$\langle \exists r.A_2 \sqsubseteq_{\mathcal{T}} \exists r.A_1, q_1 \rangle$$

also holds. By definition, we know that for all $\alpha, \beta \in [0, 1], \alpha \Rightarrow \beta \geq \beta$. It thus suffices to prove that for every model $\mathcal{I}$ of $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}, (\exists r.A_1)^{\mathcal{I}}(x) \geq q_1$.

$$
\begin{aligned}
(\exists r.A_1)^{\mathcal{I}}(x) &= \sup_{y \in \Delta^{\mathcal{I}}} \min(r^{\mathcal{I}}(x, y), A_1^{\mathcal{I}}(y)) \\
&\geq \sup_{y \in \Delta^{\mathcal{I}}, r^{\mathcal{I}}(x,y) > A_1^{\mathcal{I}}(y), A_2^{\mathcal{I}}(y) > A_1^{\mathcal{I}}(y)} A_1^{\mathcal{I}}(y) \\
&\geq q_1
\end{aligned}
$$

because $\langle A_2 \sqsubseteq_{\mathcal{T}} A_1, q_1 \rangle$ entails that $A_1^{\mathcal{I}}(x) \geq q_1$ whenever $A_2^{\mathcal{I}}(x) > A_1^{\mathcal{I}}(x)$.

From the second invariant, we have also that $\langle A \sqsubseteq_{\mathcal{T}} \exists r.A_2, q_2 \rangle$ holds. Thus, for every model $\mathcal{I}$ of $\mathcal{T}$ and every $x \in \Delta^{\mathcal{I}}$ we have that $A^{\mathcal{I}}(x) \Rightarrow$

$(\exists r.A_2)^{\mathcal{I}}(x) \geq q_2$ and $(\exists r.A_2)^{\mathcal{I}}(x) \Rightarrow (\exists r.A_1)^{\mathcal{I}}(x) \geq q_1$, which means that $A^{\mathcal{I}}(x) \Rightarrow (\exists r.A_1)^{\mathcal{I}}(x) \geq \min(q_1, q_2)$ [Exercise on transitivity]

Using again the same argument with the axiom $\langle \exists r.A_1 \sqsubseteq B, q \rangle$, we obtain that $A^{\mathcal{I}}(x) \Rightarrow B^{\mathcal{I}}(x) \geq \min(q, q_1, q_2)$ and hence $\langle A \sqsubseteq B, p \rangle$ holds.

The other two rules can be treated in a similar way [Exercise?] $\square$

When the algorithm terminates, then all the possible subsumption relations between concept names have been generated.

Lemma 2.5. *If no rule is applicable, then $\langle A_0 \sqsubseteq_{\mathcal{T}} B_0, q \rangle$ implies that there is some $q' \geq q$ such that $(B_0, q') \in S(A_0)$.*

Proof. Suppose that for every $q' \geq q$ $(B_0, q') \notin S(A_0)$. We will construct a model of $\mathcal{T}$ that violates the subsumption relation, thus showing that $\langle A_0 \not\sqsubseteq_{\mathcal{T}} B_0, q \rangle$. This model is based on the sets S and R produced by the algorithm.

- $\Delta^{\mathcal{I}} := \mathsf{N_{C}}_{\mathcal{T}} \cup \{\top\}$, (all concept names in $\mathcal{T}$, but seen as nodes)

- $B^{\mathcal{I}}(A) := \max\{q \mid (B, q) \in S(A)\}$,

- $r^{\mathcal{I}}(A, B) := \max\{q \mid (r, q) \in R(A, B)\}$.

Since $(A_0, 1) \in S(A_0)$, we have that $A_0^{\mathcal{I}}(A_0) = 1$, and by assumption, $(B_0, q') \notin S(A_0)$ for every $q' \geq q$, hence $B_0^{\mathcal{I}}(A_0) < q$. This means that $A_0^{\mathcal{I}}(A_0) \Rightarrow B_0^{\mathcal{I}}(A_0) < q$, and thus this interpretation violates the subsumption relation.

It only remains to show that $\mathcal{I}$ is indeed a model of $\mathcal{T}$. Take a GCI from $\mathcal{T}$; we have three cases:

-$\langle A_1 \sqcap A_2 \sqsubseteq B, p \rangle \in \mathcal{T}$

We need to show that, for every $A \in \Delta^{\mathcal{I}}, (A_1 \sqcap A_2)^{\mathcal{I}}(A) \Rightarrow B^{\mathcal{I}}(A) \geq p$. Let $(A_1)^{\mathcal{I}}(A) = p_1$ and $(A_2)^{\mathcal{I}}(A) = p_2$. Then $(A_1, p_1) \in S(A)$, $(A_2, p_2) \in S(A)$, and $(A_1 \sqcap A_2)^{\mathcal{I}}(A) = \min(p_1, p_2)$.

Since **R1** is not applicable, we have that there is a $p' \geq \min(p, p_1, p_2)$ such that $(B, p') \in S(A)$ and thus, $B^{\mathcal{I}}(A) \geq p'$. If $p' \geq \min(p_1, p_2)$, then $(A_1 \sqcap A_2)^{\mathcal{I}}(A) \Rightarrow B^{\mathcal{I}}(A) = 1 \geq p$. Otherwise, it follows that $p' \geq p$ and thus $(A_1 \sqcap A_2)^{\mathcal{I}}(A) \Rightarrow B^{\mathcal{I}}(A) = B^{\mathcal{I}}(A) \geq p' \geq p$.

$-\langle A_1 \sqsubseteq \exists r.B, p\rangle \in \mathcal{T}$

Let $(A_1)^{\mathcal{I}}(A) = p_1$; then $(A_1, p_1) \in S(A)$. Non applicability of **R2** yields the existence of $p' \geq \min(p, p_1)$ such that $(r, p') \in R(A, B)$. Thus $r^{\mathcal{I}}(A, B) \geq p'$ and since $B^{\mathcal{I}}(B) = 1$ we have that $(\exists r.B)^{\mathcal{I}}(A) \geq p'$. If $p' \geq p_1$, then $A_1^{\mathcal{I}}(A) \Rightarrow (\exists r.B)^{\mathcal{I}}(A) = 1 \geq p$; otherwise, $p' \geq p$ and hence $A_1^{\mathcal{I}}(A) \Rightarrow (\exists r.B)^{\mathcal{I}}(A) \geq p' \geq p$.

$-\langle \exists r.A_1 \sqsubseteq B, p\rangle \in \mathcal{T}$

Let $(\exists r.A_1)^{\mathcal{I}}(A) = p$. Since $\Delta^{\mathcal{I}}$ is finite (we consider only the concept names appearing in $\mathcal{T}$), we have that

$$p = \sup_{y \in \Delta^{\mathcal{I}}} \min(r(A, y), A_1(y)) = \max_{y \in \Delta^{\mathcal{I}}} \min(r(A, y), A_1(y))$$

and thus, there exist B', p_1, p_2 such that $r^{\mathcal{I}}(A, B') = p_1, A_1^{\mathcal{I}}(B') = p_2$, and $p = \min(p_1, p_2)$. This implies that $(r, p_1) \in R(A, B')$ and $(A_1, p_2) \in S(B')$. Since **R3** is not applicable, we have that there is a $q \geq \min(p_1, p_2)$ such that $(B, q) \in S(A)$; that is, $B^{\mathcal{I}}(A) = q \geq p$. □

A simple consequence from these lemmas is that subsumption in Gödel-$\mathcal{EL}$ w.r.t. TBoxes is decidable. One can in fact go a step further and prove that the completion algorithm works in polynomial time [Exercise?]

The pairs (A, p) used by the completion algorithm can be seen as crisp concepts that express all the individuals that belong to the concept A to a degree at least p. In this way, the completion graph is built using only crisp concepts, and so reasoning is reduced to the crisp case. We take this a step further and show that reasoning in Gödel $\mathcal{ALC}$ can be reduced to reasoning in crisp $\mathcal{ALC}$ by producing the bounding concepts and adapting the ontology accordingly.

2.4 Reasoning in Gödel $\mathcal{ALC}$

Rather than developing a new reasoning algorithm for Gödel $\mathcal{ALC}$, we will show how to transform a fuzzy ontology into a crisp one that preserves consistency. In this way, we can decide consistency of the original ontology, simply by using the known (exponential time) algorithms for crisp $\mathcal{ALC}$. For this reduction to work, we need to consider a special kind of models, called *witnessed models*.

Definition 2.6 (witnessed models). An interpretation $\mathcal{I}$ is called *witnessed* if for every concept C, role name r and $x \in \Delta^{\mathcal{I}}$ there are $y, z \in \Delta^{\mathcal{I}}$ such that

- $(\exists r.C)^{\mathcal{I}}(x) = \min(r^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y))$ and

- $(\forall r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)$

We say that an ontology is *witnessed consistent* if it has a witnessed model. △

The intuition behind witnessed interpretations is that every supremum and infimum defined by a quantifier has to be reached (or witnessed) by an element of the domain. Obviously, every witnessed model is also a model, and hence witnessed consistency implies consistency of an ontology. However, the converse does not hold.

Example 2.7. Consider the ontology $\mathcal{O}$ having only the assertion

$$\langle (\neg \forall r.A) \sqcap (\neg \exists r.\neg A)(a), 1\rangle .$$

We show that $\mathcal{O}$ has a model but no witnessed model.

Consider the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ given by:

- $\Delta^{\mathcal{I}} = \mathbb{N}$ (the set of natural numbers),

- for $n \in \mathbb{N}$, $A^{\mathcal{I}}(n) = \frac{1}{n+1}$,

- $r^{\mathcal{I}}(m, n) = 1$ iff $m = 0$, and

- $a^{\mathcal{I}} = 0$.

It then follows that

$$(\forall r.A)^{\mathcal{I}}(a^{\mathcal{I}}) = \inf_{n \in \mathbb{N}}(r^{\mathcal{I}}(0, n) \Rightarrow A^{\mathcal{I}}(n)) = \inf_{n \in \mathbb{N}} \frac{1}{n+1} = 0$$

and

$$(\exists r.\neg A)^{\mathcal{I}}(a^{\mathcal{I}}) = \sup_{n \in \mathbb{N}} \min(r^{\mathcal{I}}(0, n), (\neg A)^{\mathcal{I}}(n)) = \sup_{n \in \mathbb{N}} 0 = 0.$$

Thus, $((\neg \forall r.A) \sqcap (\neg \exists r.\neg A))^{\mathcal{I}}(a^{\mathcal{I}}) = 1$ and $\mathcal{I}$ is a model of $\mathcal{O}$.

Suppose now that $\mathcal{O}$ has a witnessed model $\mathcal{J}$. Then, there must be a witness for $(\forall r.A)(a^{\mathcal{J}})$; that is, there is an $y \in \Delta^{\mathcal{J}}$ such that $r^{\mathcal{J}}(a^{\mathcal{J}}, y) \Rightarrow A^{\mathcal{J}}(y) = 0$. This means that $r^{\mathcal{J}}(a^{\mathcal{J}}, y) > 0$ and $A^{\mathcal{J}}(y) = 0$. But then,

$$\min(r^{\mathcal{J}}(a^{\mathcal{J}}, y), (\neg A)^{\mathcal{J}}(y)) = r^{\mathcal{J}}(a^{\mathcal{J}}, y) > 0$$

but then $(\exists r.\neg A)^{\mathcal{J}}(a^{\mathcal{J}}) > 0$, which means that

$$((\neg \forall r.A) \sqcap (\neg \exists r.\neg A))^{\mathcal{J}}(a^{\mathcal{J}}) = 0$$

violating the only axiom in $\mathcal{O}$.

If we restrict reasoning to witnessed models, then we can reduce the problem to the crisp case.

Just as we have general concept inclusion axioms $C \sqsubseteq D$ in crisp DLs, one can also think of *role inclusion* axioms of the form $r \sqsubseteq s$, where r, s are two role names. An *RBox* is a finite set of role inclusions. The semantics of these axioms is the obvious one: a crisp interpretation $\mathcal{I}$ satisfies the role inclusion $r \sqsubseteq s$ if $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. It satisfies the RBox $\mathcal{R}$ if it satisfies all axioms in $\mathcal{R}$.

It is known that extending (crisp) $\mathcal{ALC}$ with role inclusions does not increase the complexity of reasoning; that is, consistency of an $\mathcal{ALC}$ ontology extended with an RBox can be decided in exponential time too. We will use role inclusion axioms to simplify our reduction from fuzzy to crisp ontologies.

Let now $\mathcal{O}$ be a fuzzy ontology. We define the set $\mathcal{V}_{\mathcal{O}}$ of membership degrees appearing in $\mathcal{O}$ as

$$\mathcal{V}_{\mathcal{O}} = \{q \in [0,1] \mid \langle \alpha, q \rangle \in \mathcal{O}\} \cup \{0, 1\}.$$

Notice that, since $\mathcal{O}$ is finite, $\mathcal{V}_{\mathcal{O}}$ must also be finite. W.l.o.g., we assume that $\mathcal{V}_{\mathcal{O}} = \{q_0, \ldots, q_n\}$ with $q_i < q_{i+1}$ for every $0 \leq i < n = |\mathcal{V}_{\mathcal{O}}|$. (That is, that $\mathcal{V}_{\mathcal{O}}$ is linearly ordered). It immediately follows that $q_0 = 0$ and $q_n = 1$.

Notice that this set is closed under the interpretation of the connectives: Gödel negation, minimum and residuum. In other words, if we apply any of these operators to elements of $\mathcal{V}_{\mathcal{O}}$, we obtain again an element of $\mathcal{V}_{\mathcal{O}}$. We will show that these are all the membership degrees that are relevant when deciding ontology consistency.

We will create n concept names $A_{q_i}, i \geq 1$ for every concept name A in $\mathcal{O}$. Intuitively, A_q represents the (crisp) set of all individuals that belong to A with a degree greater or equal to q; that is, it defines a cut on the fuzzy interpretation of A. Similarly, we introduce role names r_q for every $q \in \mathcal{V}_{\mathcal{O}}$.

Notice that we do not create the concept name A_0 nor the role name r_0, since these will be interpreted as tautological concepts and roles.

Obviously, given our interpretation of these concepts, if an individual belongs to the concept $A_{q_{i+1}}$ for some $i, 1 \leq i < n$, then it must also belong to A_{q_i}. This restriction needs to be enforced for every membership degree in $\mathcal{V}_{\mathcal{O}}$ and every concept name appearing in $\mathcal{O}$. For that reason, we introduce the TBox

$$\mathcal{T}_{\mathcal{O}} = \{A_{q_{i+1}} \sqsubseteq A_{q_i} \mid 1 \leq i < n\}.$$

A similar condition must be satisfied by the role names. We enforce this through the RBox

$$\mathcal{R}_{\mathcal{O}} = \{r_{q_{i+1}} \sqsubseteq r_{q_i} \mid 1 \leq i < n\}.$$

We define the translation ρ of complex concept expressions (C, q) for $q \in \mathcal{V}_{\mathcal{O}} \setminus \{0\}$ as follows:

- $\rho(A, q) = A_q,$

- $\rho(\top, q) = \top, \quad \rho(\bot, q) = \bot,$

- $\rho(\neg C, q) = \neg \rho(C, q_1), \quad (C \text{ has to be interpreted as } 0)$

- $\rho(C \sqcap D, q) = \rho(C, q) \sqcap \rho(D, q),$

- $\rho(C \sqcup D, q) = \rho(C, q) \sqcup \rho(D, q),$

- $\rho(\exists r.C, q) = \exists r_q.\rho(C, q),$

- $\rho(\forall r.C, q) = \displaystyle\bigsqcap_{0 < p \leq q} \forall r_p.\rho(C, p).$

Example 2.8. Let $\mathcal{V}_{\mathcal{O}} = \{0, 0.5, 0.75, 1\}$. We then have

$$
\begin{aligned}
\rho(\forall r.\neg A, 0.75) &= \forall r_{0.5}.\rho(\neg A, 0.5) \sqcap \forall r_{0.75}.\rho(\neg A, 0.75) \\
&= \forall r_{0.5}.A_{0.5} \sqcap \forall r_{0.75}.\neg\rho(A, 0.5) \\
&= \forall r_{0.5}.A_{0.5} \sqcap \forall r_{0.75}.\neg A_{0.5}.
\end{aligned}
$$

The idea is that $\rho(C, q)$ expresses all those individuals that belong to C with a degree at least q. More formally, if we have a fuzzy interpretation $\mathcal{I}$, we can transform it into a crisp interpretation $\mathcal{J}$ that satisfies this property, simply by defining, for every concept name A

$$
A_q^{\mathcal{J}} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) \geq q\}
$$

and for every role name r

$$
r_q^{\mathcal{J}} = \{(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, y) \geq q\}.
$$

Notice that $\mathcal{J}$ satisfies all the axioms in $\mathcal{T}_{\mathcal{O}} \cup \mathcal{R}_{\mathcal{O}}$.

Lemma 2.9. *Let $\mathcal{I}$ be a witnessed fuzzy interpretation and $\mathcal{J}$ the crisp interpretation constructed as above. Then, for every concept C and $q \in \mathcal{V}_{\mathcal{O}}$, it holds that $(\rho(C, q))^{\mathcal{J}} = \{x \in \Delta^{\mathcal{I}} \mid C^{\mathcal{I}}(x) \geq q\}$.*

Proof. The proof is by induction on the structure of the concept C. If C is a concept name A, then by definition $(\rho(A, q))^{\mathcal{J}} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) \geq q\}$.

Let now C be of the form $\neg D$ for some D satisfying the property and $q > 0 \in \mathcal{V}_{\mathcal{O}}$. Since we are using the Gödel negation, which always evaluates to either 0 or 1, we have that

$$
\begin{aligned}
\{x \in \Delta^{\mathcal{I}} \mid (\neg D)^{\mathcal{I}}(x) \geq q\} &= \{x \in \Delta^{\mathcal{I}} \mid (\neg D)^{\mathcal{I}}(x) = 1\} \\
&= \{x \in \Delta^{\mathcal{I}} \mid D^{\mathcal{I}}(x) = 0\}.
\end{aligned}
$$

On the other hand,

$$
\rho(\neg D, q)^{\mathcal{J}} = \neg\rho(D, q_1) = \Delta^{\mathcal{I}} \setminus \{x \in \Delta^{\mathcal{I}} \mid D^{\mathcal{I}}(x) > 0\}.
$$

For C of the form $\forall r.D$, let x be such that $(\forall r.D)^{\mathcal{I}}(x) \geq q$, and $p, 0 < p \leq q$. For every $y \in \Delta^{\mathcal{I}}$ if $(x, y) \in r_p^{\mathcal{J}}$, then $r^{\mathcal{I}}(x, y) \geq p$

and hence $D^{\mathcal{I}}(y) \geq p$; by induction, it follows that $y \in \rho(D, p)$. This means that $x \in \forall r_p.\rho(D, p)$. As this is true for every $p, 0 < p \leq q$, $x \in \rho(\forall r.D, q)$. Conversely, if $(\forall r.D)^{\mathcal{I}}(x) < q$, then there is a $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) \Rightarrow D^{\mathcal{I}}(y) < q$. In particular, this means that $r^{\mathcal{I}}(x, y) > D^{\mathcal{I}}(y)$ and $D^{\mathcal{I}}(y) < q$. Thus, there is a $p \leq q$ such that $r^{\mathcal{I}}(x, y) \geq p > D^{\mathcal{I}}(y)$; that is, $(x, y) \in r_p^{\mathcal{J}}$ but $y \notin \rho(D, p)^{\mathcal{J}}$, hence $x \notin \forall r_p.\rho(D, p)$, and in particular $x \notin \rho(\forall r.D, q)$.

The proofs of the other constructors follow the same lines. $\square$

Conversely, if we have a crisp model $\mathcal{J}$ of $\mathcal{T}_{\mathcal{O}} \cup \mathcal{R}_{\mathcal{O}}$, we can construct a fuzzy interpretation $\mathcal{I}$ that sets, for every concept name A and role name r

$$
A^{\mathcal{I}}(x) = \max\{q \in \mathcal{V}_{\mathcal{O}} \mid x \in A_q^{\mathcal{J}}\} \quad \text{(0 if in none)}
$$

$$
r^{\mathcal{I}}(x, y) = \max\{q \in \mathcal{V}_{\mathcal{O}} \mid (x, y) \in r_q^{\mathcal{J}}\}.
$$

Notice that this interpretation is witnessed.

Lemma 2.10. *Let $\mathcal{J}$ be a crisp model of $\mathcal{T}_{\mathcal{O}} \cup \mathcal{R}_{\mathcal{O}}$ and $\mathcal{I}$ the fuzzy interpretation constructed as above. Then, for every concept C and $q \in \mathcal{V}_{\mathcal{O}}$, it holds that $(\rho(C, q))^{\mathcal{J}} = \{x \in \Delta^{\mathcal{I}} \mid C^{\mathcal{I}}(x) \geq q\}$.*

Proof. The proof is by induction, following similar ideas as Lemma 2.9. Students should try it by themselves. $\square$

These two lemmas together express that the translation from fuzzy to crisp concepts preservers the properties of every interpretation, and adds no new ones. We will use this to show that the ontology we will now construct has a model if and only if the original fuzzy ontology has a witnessed model.

Now, we need only translate the axioms. We define the translation κ, transforming fuzzy axioms into sets of crisp axioms, as follows:

- $\kappa(\langle C(a), q \rangle) = \{\rho(C, q)(a)\}$,

- $\kappa(\langle r(a, b), q \rangle) = \{r_q(a, b)\}$,

- $\kappa(\langle C \sqsubseteq D, q \rangle) = \{\rho(C, p) \sqsubseteq \rho(D, p) \mid 0 < p \leq q\}$.

We extend this translation to fuzzy ontologies as follows

$$
\kappa(\mathcal{O}) := \cup_{\alpha \in \mathcal{O}} \kappa(\alpha) \cup \mathcal{T}_{\mathcal{O}} \cup \mathcal{R}_{\mathcal{O}}.
$$

Theorem 2.11. *A fuzzy ontology $\mathcal{O}$ is witnessed consistent iff the crisp ontology $\kappa(\mathcal{O})$ is consistent.*

Proof. Suppose $\mathcal{O}$ is consistent. Then it has a fuzzy model $\mathcal{I}$. Consider the crisp interpretation $\mathcal{J}$ constructed as for Lemma 2.9 and let $a^{\mathcal{J}} = a^{\mathcal{I}}$ for every concept name a. $\mathcal{J}$ is a model of $\mathcal{T}_{\mathcal{O}} \cup \mathcal{R}_{\mathcal{O}}$; hence, it remains only to show that it satisfies all the axioms in $\kappa(\alpha)$ for every $\alpha \in \mathcal{O}$.

For an assertion of the form $\langle C(a), q \rangle \in \mathcal{O}$, since $\mathcal{I}$ is a model of $\mathcal{O}$, we know that $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq q$, and thus, $a^{\mathcal{J}} = a^{\mathcal{I}} \in \rho(C, q)^{\mathcal{J}}$ (Lemma 2.9); thus $\mathcal{J}$ satisfies $\kappa(\langle C(a), q \rangle)$. The proof for role assertions is analogous.

For an axiom $\langle C \sqsubseteq D, q \rangle \in \mathcal{O}$, let $p, 0 < p \leq q$ and $x \in \rho(C, p)^{\mathcal{J}}$. This means that $C^{\mathcal{I}}(x) \geq p$ (Lemma 2.9) and hence, since $\mathcal{I}$ is a model of $\mathcal{O}$, it follows that $D^{\mathcal{I}}(x) \geq p$. But then, $x \in \rho(D, p)^{\mathcal{J}}$ (Lemma 2.9).

Thus, $\mathcal{J}$ is a model of $\kappa(\mathcal{O})$, and $\kappa(\mathcal{O})$ is consistent.

For the converse, that is, for proving that if $\kappa(\mathcal{O})$ is consistent, then $\mathcal{O}$ is also consistent, the argument is analogous, using the construction and result from Lemma 2.10. $\qquad \square$

This shows that consistency of fuzzy $\mathcal{ALC}$ ontologies under Gödel semantics is decidable. Moreover, it can be decided in time exponential on the size of $\kappa(\mathcal{O})$. We analyse now how big this crisp ontology is.

Notice first that the number of axioms in $\kappa(\mathcal{O})$ is bounded by $|\mathcal{V}_{\mathcal{O}}| \cdot (|\mathcal{O}| + 2)$. The concepts appearing in each axiom in $\kappa(\mathcal{O})$ are transformations from the concepts in $\mathcal{O}$. In the worst case, we have a concept C that is a nesting of value restrictions with a very high degree. In that case, the size of $\rho(C, 1)$ is bounded by $|\mathcal{V}_{\mathcal{O}}| \cdot m$ where m is the maximal nesting of value restrictions in a concept. But both $|\mathcal{V}_{\mathcal{O}}|$ and m are bounded linearly by the size of $\mathcal{O}$, and hence in total, the size of $\kappa(\mathcal{O})$ is bounded cubically by the size of $\mathcal{O}$. Since deciding consistency of $\kappa(\mathcal{O})$ is exponential in the size of $\kappa(\mathcal{O})$, this yields also an exponential upper bound for deciding consistency of $\mathcal{O}$, which is optimal.

Question for the careful reader: where were the properties of witnessed models used?

3 Fuzzy Description Logics with General t-norms

Gödel semantics are a simple generalization of the crisp semantics to the interval $[0, 1]$. As we have seen, this extension preserves most of the properties of crisp DLs; most notably, the complexity of reasoning with it. Moreover, the logic produced is "simple" in the sense that it is not hard to adapt existing reasoning methods for crisp DLs to the Gödel semantics. Two methods that we presented were: building a new algorithm based on the ideas for crisp reasoning ($\mathcal{EL}$), and reducing fuzzy reasoning to the crisp case ($\mathcal{ALC}$).

In reality, the operators that define the Gödel semantics are just a special case of a family of fuzzy operators known as *triangular norms* (t-norms). Each member of this family defines a semantics for fuzzy DLs.

3.1 Triangular Norms

Before we define t-norms in detail, we motivate our choices by describing some properties that the interpretation of the different constructors should satisfy. First, since we want a logic in which the values of complex concepts at a given individual x depend only on the values of the concept names and role names associated to x, we require every constructor to be *truth functional*.

Formally, an n-ary connective c is truth functional if there is a function $f_c : [0, 1]^n \to [0, 1]$ such that, for any n concepts $C_1, \ldots, C_n$, the membership degree of the formula $c(C_1, \ldots, C_n)$ is obtained by applying f_c to the membership degrees of $C_1, \ldots, C_n$. This function f_c is called the *interpretation function* of c.

For example, consider the conjunction constructor $\sqcap$. The semantics of fuzzy DL should be such that $(C \sqcap D)^{\mathcal{I}}(x) = f_{\sqcap}(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$ for

every interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$.

The second condition is that the interpretation function of every constructor generalizes the crisp semantics of DL. This means that if the subconcepts are interpreted to the values 0 or 1, then the interpretation function should behave as the crisp constructor. Hence, for example, if $C^{\mathcal{I}}(x) = 1 = D^{\mathcal{I}}(x)$, then $(C \sqcap D)^{\mathcal{I}}(x) = 1$ or, in general, if $C^{\mathcal{I}}(x), D^{\mathcal{I}}(x) \in \{0, 1\}$, then $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \wedge D^{\mathcal{I}}(x)$, where $\wedge$ denotes the classical conjunction operator.

It is easy to see that the Gödel semantics satisfy these two properties for all the connectives in $\mathcal{ALC}$. We will show that there are many other functions that also satisfy it.

We first give the general description of the interpretation functions for conjunction: that they describe a continuous t-norm. Later we will show that all the other connectives can be constructed from conjunction in a unique and adequate way.

Definition 3.1. A *t-norm* is an associative and commutative binary operator $\otimes$ on $[0, 1]$ that is non-decreasing in both arguments and has 1 as its identity element.

In other words, $\otimes$ must satisfy the following conditions:

1. if $x_1 \le x_2$ and $y_1 \le y_2$, then $x_1 \otimes y_1 \le x_2 \otimes y_2$ (non-decreasing), and

2. $1 \otimes x = x$ for all $x \in [0, 1]$ (unit)

The t-norm $\otimes$ is called *continuous* if it is a continuous function (in the usual analysis sense). $\triangle$

The intuition behind the use of t-norms for fuzzy logics is the following. If an element has a high membership degree to a conjunction, then it must belong to each of the conjuncts with a high degree too (non-decreasing); additionally, the order of the conjuncts should not influence the degree (commutativity). The other conditions follow from generalizing classical conjunction.

Three important t-norms are:

1. *Gödel t-norm*: $x \otimes y = \min\{x, y\}$,

2. *Łukasiewicz t-norm*: $x \otimes y = \max\{x + y - 1, 0\}$,

3. *Product t-norm*: $x \otimes y = x \cdot y$.

[Exercise: show that they are t-norms]

For every continuous t-norm $\otimes$, there exists a unique binary operator $\Rightarrow$ (called the *residuum* of $\otimes$) such that for every $x, y, z \in [0, 1]$,

$$z \le x \Rightarrow y \text{ iff } x \otimes z \le y.$$

This operator is defined as $x \Rightarrow y := \max\{z \mid x \otimes z \le y\}$. [Exercise: prove this]

In this chapter we will consider only continuous t-norms. Thus, in the following, whenever we say "t-norm" we mean "continuous t-norm".

Some of the properties of residua, which have been shown to hold already for the Gödel t-norm, and for the crisp semantics, are the following:

Exercise. Show that for every continuous t-norm $\otimes$ and its residuum $\Rightarrow$, and every $x, y, z \in [0, 1]$

1. $x \le y$ iff $(x \Rightarrow y) = 1$,

2. $(1 \Rightarrow x) = x$,

3. $x \Rightarrow (y \Rightarrow z) = (x \otimes y) \Rightarrow z$.

Proposition 3.2. *The following operators define the residua of the three main t-norms: for $x > y$,*

1. *Łukasiewicz implication: $x \Rightarrow y = 1 - x + y$*

2. *Product implication: $x \Rightarrow y = y/x$ (also called Goguen implication)*

3. *Gödel implication: $x \Rightarrow y = y$.*

and $x \Rightarrow y = 1$ if $x \le y$.

[Note: for $x \le y$ we know, from the previous exercise, that $x \Rightarrow y = 1$ always]

Proof. Let $1 \ge x > y$; then

1. $x \otimes z \leq y$ iff $x + z - 1 \leq y$ iff $z \leq 1 - x + y$; thus $1 - x + y = \max\{z \mid x \otimes z \leq y\}$.

2. $x \otimes z \leq y$ iff $x \cdot z \leq y$ iff $z \leq y/x$.

3. $x \otimes z \leq y$ iff $\min\{x, z\} \leq y$ iff $z \leq y$.

$\square$

Notice that the residuum is antimonotonic on the first parameter and monotonic on the second. In other words, if $x \leq x'$, then $x \Rightarrow y \geq x' \Rightarrow y$ and $y \Rightarrow x \leq y \Rightarrow x'$.

The residuum defines the unary *precomplement* $\ominus$ given by $\ominus(x) := x \Rightarrow 0$ for all $x \in [0, 1]$. This operator is used for interpreting the negation constructor.

The precomplement of the Gödel and product t-norms is the Gödel negation:

$$\ominus(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for the Łukasiewicz t-norm is the *involutive negation* $\ominus(x) = 1 - x$.

Finally, to interpret the disjunction constructor, we use the *t-conorm* $\oplus$ defined as: $x \oplus y = 1 - (1 - x) \otimes (1 - y)$ for every $x, y \in [0, 1]$; that is, we define an operator that satisfies the De Morgan rules with the interpretation of the conjunction $\otimes$.

The t-conorms of the three main t-norms defined above are the following:

1. *Gödel t-conorm*: $x \otimes y = \max\{x, y\}$,

2. *Łukasiewicz t-conorm*: $x \otimes y = \min\{x + y, 1\}$,

3. *Product t-conorm*: $x \otimes y = x + y - x \cdot y$.

The operators defined by the Łukasiewicz t-norm are a popular choice in fuzzy logics, since some of the common equivalences hold. [Proof as exercise? Is very very simple]

Lemma 3.3. *If $\otimes$ is the Łukasiewicz t-norm, then the following equivalences hold for every $x, y \in [0, 1]$:*

- $\ominus \ominus x = x$,

- $x \Rightarrow y = \ominus x \oplus y$,

- $x \oplus y = \ominus(\ominus x \otimes \ominus y)$.

However, these equivalences do not hold for the Gödel nor the product t-norms. For example, let $x = 0.5, y = 0.5$. Then

$$\begin{aligned}
\ominus \ominus x = \ominus 0 = 1 &\neq 0.5 = x \\
x \Rightarrow y = 1 &\neq 0.5 = 0 \oplus 0.5 = \ominus x \oplus y \\
x \oplus y &< 1 = \ominus 0 = \ominus(\ominus x \otimes \ominus y).
\end{aligned}$$

There exist infinitely many continuous t-norms.

A way to construct new t-norms is to combine previously known ones using the *ordinal sum*.

Definition 3.4. Consider a (possibly infinite) sequence of elements

$$0 = a_1 < a_2 < \cdots < a_n < a_{n+1} = 1.$$

and t-norms $\otimes_1, \ldots, \otimes_n$.

The *ordinal sum* of $(a_i, \otimes_i)_{1 \leq i \leq n}$ is the binary operator $\otimes$ defined for every $x, y \in [0, 1]$ as:

–if $x, y \in [a_i, a_{i+1}]$, then $x = a_i + (a_{i+1} - a_i)x'$, $\quad y = a_i + (a_{i+1} - a_i)y'$, for some $x', y' \in [0, 1]$. Then,

$$x \otimes y = a_i + (a_{i+1} - a_i)(x' \otimes_i y')$$

–if x, y belong to different intervals, then $x \otimes y = \min(x, y)$. $\quad \triangle$

That is, at each interval, we use a (scaled and relocated) version of a t-norm. [small drawing of unit interval, cut in pieces]

It is easy to see that the ordinal sum $\otimes$ of continuous t-norms is itself a continuous t-norm, and if $n > 1$, and $\otimes_i$ is not the Gödel t-norm for some $i, 1 \leq i \leq n$, then $\otimes$ is different from all the t-norms $\otimes_1, \ldots, \otimes_n$ used in the construction.

For example, we can consider the t-norm $\otimes$ built by putting the product t-norm in the intervals $[0, 0.5], [0.5, 1]$. This t-norm is defined by:

$$x \otimes y = \begin{cases} 2x \cdot y & \text{if } x, y \in [0, 0.5] \\ 2(x - 0.5)(y - 0.5) + 0.5 & \text{if } x, y \in [0.5, 1] \\ \min(x, y) & \text{otherwise.} \end{cases}$$

That the converse holds is trivial: every t-norm $\otimes$ is the ordinal sum of itself over the whole interval $[0, 1]$ (that is, $a_2 = 1$).

However, it is possible to prove a much stronger results: every continuous t-norm can be expressed as the ordinal sum of the Łukasiewicz, product and Gödel t-norms. This is known as the *Mostert-Shields Theorem*.

For a t-norm formed as the ordinal sum of t-norms as described above, the residuum $\Rightarrow$ is given for every $x > y \in [0, 1]$ by:

–if $x, y \in [a_i, a_{i+1}]$, then $x = a_i + (a_{i+1} - a_i)x'$, $\quad y = a_i + (a_{i+1} - a_i)y'$, $x, y \in [0, 1]$. We then set

$$x \Rightarrow y = a_i + (a_{i+1} - a_i)x' \Rightarrow_i y'$$

–if x, y belong to different intervals, then $x \Rightarrow y = y$
where $\Rightarrow_i$ represents the residuum of the t-norm $\otimes_i$.

Once again, if $x \leq y$, then $x \Rightarrow y = 1$.

For example, the residuum of the t-norm built from the product t-norm in the intervals $[0, 0.5], [0.5, 1]$ is:

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y/2x & \text{if } x, y \in [0, 0.5], \\ (y - 0.5)/(2(x - 0.5)) + 0.5 & \text{if } x, y \in [0.5, 1], \\ y & \text{otherwise} \end{cases}$$

With this, we are ready to define the general semantics for fuzzy $\mathcal{ALC}$.

3.2 $\otimes\text{-}\mathcal{ALC}$

Every t-norm $\otimes$ defines a fuzzy logic $\otimes\text{-}\mathcal{ALC}$. The syntax of this logic, and its axioms, is exactly the same as for Gödel $\mathcal{ALC}$ (which is, in fact,

an instance of $\otimes\text{-}\mathcal{ALC}$, where $\otimes$ is the Gödel t-norm). The difference relies on how complex concepts are interpreted, and which kinds of interpretations define a model.

Definition 3.5 (semantics of $\otimes\text{-}\mathcal{ALC}$). A *fuzzy interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called the *domain* and $\cdot^{\mathcal{I}}$ is a function mapping:

- every concept name $A \in \mathsf{N_C}$ to a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$, and

- every role name $r \in \mathsf{N_R}$ to a function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$.

This function is extended to concept descriptions as follows. For every $x \in \Delta^{\mathcal{I}}$:

- $\top^{\mathcal{I}}(x) = 1$, $\bot^{\mathcal{I}}(x) = 0$,

- $(\neg C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$,

- $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$,

- $(C \sqcup D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$,

- $(\exists r.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)\}$,

- $(\forall r.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)\}$.

$\mathcal{I}$ *satisfies* the axiom $\langle C(a), q \rangle$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq q$; it *satisfies* $\langle r(a, b), q \rangle$ if $r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq q$; and it *satisfies* the GCI $\langle C \sqsubseteq D, q \rangle$ if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq q$ for every $x \in \Delta^{\mathcal{I}}$.

$\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$. $\mathcal{O}$ is *consistent* if it has a model. $\triangle$

It is important to emphasize that the semantics depend on the t-norm chosen. Consider for example the ABox $\{\langle A(a), 0.1 \rangle, \langle (\neg A)(a), 0.1 \rangle\}$. Under the Łukasiewicz t-norm, this ontology is consistent: the interpretation $\mathcal{I} = (\{a\}, \cdot)$ with $A^{\mathcal{I}}(a) = 0.9$ satisfies both axioms since

$$(\neg A)^{\mathcal{I}}(a) = A^{\mathcal{I}}(a) \Rightarrow 0 = 1 - 0.9 = 0.1$$

However, if we use the Gödel or product semantics, at least one of A or $\neg A$ is interpreted as 0 (Gödel negation!), and hence if $\mathcal{I}$ satisfies the

first axiom, i.e., if $A^{\mathcal{I}}(a) \geq 0.1$, it must be the case that $(\neg A)^{\mathcal{I}}(a) = 0$, which violates the second axiom.

In general, only a few logical equivalences between constructors can be guaranteed. In fact, as we have seen before, in general

$$\neg\neg C \not\equiv C, \quad C \sqcup \neg C \not\equiv \top, \quad \exists r.C \not\equiv \neg\forall r.(\neg C), \quad \neg\forall r.C \not\equiv \exists r.(\neg C),$$
$$\neg(C \sqcup D) \not\equiv \neg C \sqcap \neg D, \quad \neg(C \sqcap D) \not\equiv \neg C \sqcup \neg D$$

[Exercise: prove the last two. Hint: you need a "combined" t-norm (not one of the basic three)– prove that the equivalences hold in the L, prod and G t-norms!]

Lemma 3.6. *For every t-norm $\otimes$, the following equivalences hold:*

- $C \sqcap \neg C \equiv \bot,$

- $\neg\exists r.C \equiv \forall r.(\neg C).$

Proof. • Recall that $x \Rightarrow 0 = \max\{z \mid x \otimes z \leq 0\}$, and hence $x \otimes (x \Rightarrow 0) \leq 0$. For an interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$ we have

$$(C \sqcap \neg C)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes (C^{\mathcal{I}}(x) \Rightarrow 0) = 0.$$

- For the other equivalence, we have

$$\begin{aligned}
(\forall r.\neg C)^{\mathcal{I}}(x) &= \inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \Rightarrow (C^{\mathcal{I}}(y) \Rightarrow 0)\} \\
&= \inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) \Rightarrow 0\} \\
&= \sup_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\} \Rightarrow 0 = (\neg\exists r.C)^{\mathcal{I}}(x)\square
\end{aligned}$$

Other properties hold for specific t-norms only. As we have seen, in the Łukasiewicz t-norm, many of the standard equivalences of classical logic hold, which gives rise to interdefinability of $\exists$ and $\forall$, or the law of excluded middle.

Lemma 3.7. *Under the Łukasiewicz semantics, the following equivalences hold:*

- $C \sqcup \neg C \equiv \top,$

- $\neg\forall r.C \equiv \exists r.(\neg C).$

Proof. $C^{\mathcal{I}}(x) \oplus (1 - C^{\mathcal{I}}(x)) = \min\{C^{\mathcal{I}}(x) + 1 - C^{\mathcal{I}}(x), 1\} = 1.$

$$\begin{aligned}
1 - \inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\} &= \sup_{y \in \Delta^{\mathcal{I}}}\{1 - r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\} \\
&= \sup_{r^{\mathcal{I}}(x,y) > C^{\mathcal{I}}(y)}\{1 - (1 - r^{\mathcal{I}}(x,y) + C^{\mathcal{I}}(y))\} \\
&= \sup_{r^{\mathcal{I}}(x,y) > C^{\mathcal{I}}(y)}\{r^{\mathcal{I}}(x,y) - C^{\mathcal{I}}(y)\} \\
&= \sup_{r^{\mathcal{I}}(x,y) > C^{\mathcal{I}}(y)}\{r^{\mathcal{I}}(x,y) \otimes (\neg C)^{\mathcal{I}}(y)\}. \quad \square
\end{aligned}$$

In the Gödel t-norm, we also have that conjunction and disjunction are idempotent; that is, $C \sqcap C \equiv C, C \sqcup C \equiv C$. This does not hold in any other continuous t-norm.

We have shown in the previous chapter that if we use the Gödel t-norm for the semantics, then the complexity of reasoning does not increase. This unfortunately does not hold for general t-norms. In fact, we will show that if we choose the Łukasiewicz semantics, then ontology consistency becomes an undecidable problem.

3.3 Undecidability of Ł-$\mathcal{ALC}$

We show undecidability of our problem by a reduction from a known undecidable problem. The problem we choose in this case is the *Post Correspondence Problem*.

Definition 3.8. Let $(v_1, w_1), \ldots, (v_n, w_n)$ be a sequence of pairs of words from an alphabet Σ. A finite chain of indices $i_1 \ldots i_k$, $k \geq 1$, where $i_j \in \{1, \ldots, n\}$ is called a *solution* if $v_{i_1} v_{i_2} \cdots v_{i_k} = w_{i_1} w_{i_2} \cdots w_{i_k}$. The *Post Correspondence Problem* (PCP) consists on deciding whether there is a solution or not.

For example, consider the pairs $(v_1, w_1) = (a, ab), (v_2, w_2) = (ba, a)$. The chain 12 is a solution to this instance, since $v_1 v_2 = aba = w_1 w_2$.

On the other hand, the instance formed by the pairs $(v_1, w_1) = (a, b)$, $(v_2, w_2) = (b, a)$ has no solution.

Before we present the reduction, we introduce some helpful abbreviations.

First, since the alphabet Σ is finite, we can assume w.l.o.g. that $\Sigma = \{1, \ldots, s\}$ (the first s natural numbers) with $s > 1$. Under this assumption, we can see every word in Σ^+ as a natural number in base $s+1$, where 0 never appears in this expression. Using this intuition, we will represent the empty word ε as the number 0.

For a word $u = u_1 \cdots u_m \in \Sigma^*$, where $u_i \in \Sigma$ for all $i, 1 \leq i \leq m$, $\overleftarrow{u}$ denotes the word $\overleftarrow{u} = u_m u_{m-1} \cdots u_1$.

Given a sequence $\nu = i_1 \cdots i_k \in \{1, \ldots, n\}^*$, we will denote as v_ν the word $v_{i_1} \cdots v_{i_k}$ and $w_\nu = w_{i_1} \cdots w_{i_k}$.

Finally, the expression nC abbreviates the n-ary disjunction of the concept C with itself. Formally, $1C = C$ and $(n+1)C = C \sqcup nC$. Under the Łukasiewicz semantics, we have that $(nC)^{\mathcal{I}}(x) = \min\{n(C^{\mathcal{I}}(x)), 1\}$ for all $\mathcal{I}, x \in \Delta^{\mathcal{I}}$.

Let now $\mathcal{P}$ an instance of the PCP given by $(v_1, w_1), \ldots, (v_n, w_n)$. We can visualize $\mathcal{P}$ as a tree, with nodes given by each word $\nu \in \{1, \ldots, n\}$ and where each node is labeled by the two words v_ν and w_ν. [Hand-drawn tree]

To decide whether $\mathcal{P}$ has a solution is equivalent to deciding whether there is a node in this tree where the two words it is labeled with are equal; that is, $v_\nu = w_\nu$.

The idea of the reduction is to create an ontology $\mathcal{O}_\mathcal{P}$ that only accepts models that "include" this tree (or a representation of it) and further enforce that $v_\nu \neq w_\nu$ for every node (except the root node). It then follows that $\mathcal{P}$ has a solution iff $\mathcal{O}_\mathcal{P}$ has no models, i.e., is inconsistent. For this reduction we will once again limit reasoning to witnessed models.

For the reduction, we will encode each word $w \in \Sigma^*$ as the number $0.\overleftarrow{w} \in [0, 1]$ (in base $s+1$). For instance, the word 112 is encoded as 0.211 and the empty word is encoded as 0. Since we consider only finite words, all the encodings are always strictly smaller than 1. We will also use two designated concept names V and W, whose interpretation at a node $\nu \in \{1, \ldots, n\}^*$ will correspond to the encoding of v_ν and w_ν, respectively.

We follow an inductive approach to simulate the search tree within the models of the ontology $\mathcal{O}_\mathcal{P}$.

First, we want to represent the root of the tree. That is, we want an element δ_ε such that $V^{\mathcal{I}}(\delta_\varepsilon) = W^{\mathcal{I}}(\delta_\varepsilon) = 0$, which is the encoding of the empty word. This is ensured by the two assertions

$$\mathcal{A}_\mathcal{P}^0 := \{\langle(\neg V)(a), 1\rangle, \langle(\neg W)(a), 1\rangle\}.$$

Obviously, every interpretation satisfying these axioms will set $V^{\mathcal{I}}(a^{\mathcal{I}}) = 0$ and likewise for W.

Let now $\delta \in \Delta^{\mathcal{I}}$ be a node encoding the words v, w; that is, $V^{\mathcal{I}}(\delta) = 0.\overleftarrow{v}$ and $W^{\mathcal{I}}(\delta) = 0.\overleftarrow{w}$. We want to ensure that, for every $1 \leq i \leq n$, there is a node γ that encodes the words vv_i and ww_i. Let $1 \leq i \leq n$ and assume that we have two concept names V_i, W_i with $V_i^{\mathcal{I}}(\delta) = 0.\overleftarrow{v_i}$ and $W_i^{\mathcal{I}}(\delta) = 0.\overleftarrow{w_i}$. We define the TBox

$$
\begin{aligned}
\mathcal{T}_\mathcal{P}^i \quad := \quad & \{\langle \top \sqsubseteq \exists r_i.\top, 1\rangle && (3.1)\\
& \left\langle (s+1)^{|v_i|} V_i' \sqsubseteq V, 1\right\rangle, \left\langle V \sqsubseteq (s+1)^{|v_i|} V_i', 1\right\rangle && (3.2)\\
& \langle \exists r_i.V \sqsubseteq V_i' \sqcup V_i, 1\rangle, \langle V_i' \sqcup V_i \sqsubseteq \forall r_i.V, 1\rangle && (3.3)\\
& \left\langle (s+1)^{|w_i|} W_i' \sqsubseteq W, 1\right\rangle, \left\langle W \sqsubseteq (s+1)^{|w_i|} W_i', 1\right\rangle &&\\
& \langle \exists r_i.W \sqsubseteq W_i' \sqcup W_i, 1\rangle, \langle W_i' \sqcup W_i \sqsubseteq \forall r_i.W, 1\rangle\}.
\end{aligned}
$$

The first axiom ensures that $(\exists r_i.\top)^{\mathcal{I}}(\delta) = 1$. Since we are dealing with witnessed models, this means that there exists a $\gamma \in \Delta^{\mathcal{I}}$ such that $r_i^{\mathcal{I}}(\delta, \gamma) \otimes \top^{\mathcal{I}}(\gamma) = 1$, and hence $r_i^{\mathcal{I}}(\delta, \gamma) = 1$.

The axioms from (3.2) state that $((s+1)^{|v_i|} V_i')^{\mathcal{I}}(\delta) = V^{\mathcal{I}}(\delta) < 1$, hence we have that $(s+1)^{|v_i|}(V_i'^{\mathcal{I}}(\delta)) = 0.\overleftarrow{v}$, which means that $V_i'^{\mathcal{I}}(\delta) = 0.0 \ldots 0\overleftarrow{v}$ [$|v_i|$ zeroes].

From this it follows that $(V_i' \sqcup V_i)^{\mathcal{I}}(\delta) = 0.\overleftarrow{v_i} + 0.0 \ldots 0\overleftarrow{v} = 0.\overleftarrow{vv_i}$.

We now look at the axioms from (3.3). The first one says that

$$
\begin{aligned}
0.\overleftarrow{vv_i} \quad = \quad & (V_i' \sqcup V_i)^{\mathcal{I}}(\delta) \geq (\exists r_i.V)^{\mathcal{I}}(\delta)\\
\geq \quad & \sup_{\eta \in \Delta^{\mathcal{I}}} r_i^{\mathcal{I}}(\delta, \eta) \otimes V^{\mathcal{I}}(\eta)\\
\geq \quad & r_i^{\mathcal{I}}(\delta, \gamma) \otimes V^{\mathcal{I}}(\gamma) = V^{\mathcal{I}}(\gamma).
\end{aligned}
$$

From the second, it follows that

$$
\begin{aligned}
0.\overleftarrow{vv_i} &= (V_i' \sqcup V_i)^{\mathcal{I}}(\delta) \leq (\forall r_i.V)^{\mathcal{I}}(\delta) \\
&\leq \inf_{\eta \in \Delta^{\mathcal{I}}} r_i^{\mathcal{I}}(\delta, \eta) \Rightarrow V^{\mathcal{I}}(\eta) \\
&\leq r_i^{\mathcal{I}}(\delta, \gamma) \Rightarrow V^{\mathcal{I}}(\gamma) = V^{\mathcal{I}}(\gamma).
\end{aligned}
$$

Thus, together they express that $V^{\mathcal{I}}(\gamma) = 0.\overleftarrow{vv_i}$ as desired.

Using the same argument, the last four axioms state that $W^{\mathcal{I}}(\gamma) = 0.\overleftarrow{ww_i}$.

Now, recall that this argument depends on having $V_i^{\mathcal{I}}(\delta) = 0.\overleftarrow{v_i}$. Thus, in order to build the tree inductively, we need to ensure that this holds at every node of the tree (and likewise for W_i) and for every $i, 1 \leq i \leq n$. For this reason, we consider additionally the TBox

$$
\begin{aligned}
\mathcal{T}_{\mathcal{P}}^0 \quad := \quad &\{ \langle \top \sqsubseteq V_i, 0.\overleftarrow{v_i} \rangle, \langle \top \sqsubseteq \neg V_i, 1 - 0.\overleftarrow{v_i} \rangle \\
&\langle \top \sqsubseteq W_i, 0.\overleftarrow{w_i} \rangle, \langle \top \sqsubseteq \neg W_i, 1 - 0.\overleftarrow{w_i} \rangle \mid 1 \leq i \leq n \},
\end{aligned}
$$

that ensures that V_i and W_i are interpreted as the constants $0.\overleftarrow{v_i}$ and $0.\overleftarrow{w_i}$ over all the elements of the domain.

We will also consider a bound on the difference between two words, that will be helpful for deciding whether $\mathcal{P}$ has a solution or not. Let $k := \max\{|u| \mid u \in \{v_i, w_i \mid 1 \leq i \leq n\}\}$ be the maximal length of a word in the instance $\mathcal{P}$.

Notice that for every $\nu \in \{1, \ldots, n\}^*$, $|v_\nu| \leq k|\nu|$ and $|w_\nu| \leq k|\nu|$. Thus, if $v_\nu \neq w_\nu$, $0.\overleftarrow{v_\nu}$ and $0.\overleftarrow{w_\nu}$ must differ in one of the first $k|\nu|$ digits.

If $\overleftarrow{v_\nu} > \overleftarrow{w_\nu}$, then $1 - 0.\overleftarrow{v_\nu} + 0.\overleftarrow{w_\nu} \leq 1 - (s+1)^{-k|\nu|} = 0.9 \ldots 9$ [$k|\nu|$ nines].

If $\overleftarrow{w_\nu} > \overleftarrow{v_\nu}$, then $1 - 0.\overleftarrow{w_\nu} + 0.\overleftarrow{v_\nu} \leq 1 - (s+1)^{-k|\nu|} = 0.9 \ldots 9$.

We will use an additional concept name M to keep this bound. For technical reasons, we will make the bound a little larger, although this will not make any problem. Thus, we define the ontology

$$
\begin{aligned}
\mathcal{O}_0 \quad := \quad &\{ \langle M(a), 0.9 \rangle, \langle \neg M(a), 0.1 \rangle \\
&\left\langle (s+1)^k M' \sqsubseteq \neg M, 1 \right\rangle, \left\langle \neg M \sqsubseteq (s+1)^k M', 1 \right\rangle \} \\
\cup \quad &\{ \langle \exists r_i.\neg M \sqsubseteq M', 1 \rangle, \langle M' \sqsubseteq \forall r_i.\neg M, 1 \rangle \mid 1 \leq i \leq n \}.
\end{aligned}
$$

We can then combine all these ontologies together to construct

$$
\mathcal{O}_{\mathcal{P}} := \mathcal{A}_{\mathcal{P}}^0 \cup \mathcal{O}_0 \cup \{ \mathcal{T}_{\mathcal{P}}^i \mid 0 \leq i \leq n \}.
$$

As described before, the intuition is that every model of this ontology should describe the search tree for $\mathcal{P}$. We first give a "canonical" description of this tree through the interpretation $\mathcal{I}_{\mathcal{P}} = (\{1, \ldots, n\}^*, \cdot^{\mathcal{I}_{\mathcal{P}}})$ given by:

- $a^{\mathcal{I}_{\mathcal{P}}} = \varepsilon$,

- $V^{\mathcal{I}_{\mathcal{P}}}(\mu) = 0.\overleftarrow{v_\mu}$, $W^{\mathcal{I}_{\mathcal{P}}}(\mu) = 0.\overleftarrow{w_\mu}$,

- $V_i^{\mathcal{I}_{\mathcal{P}}}(\mu) = 0.\overleftarrow{v_i}$, $W_i^{\mathcal{I}_{\mathcal{P}}}(\mu) = 0.\overleftarrow{w_i}$,

- $M^{\mathcal{I}_{\mathcal{P}}}(\mu) = 1 - (s+1)^{k|\mu|+1}$,

- $r_i^{\mathcal{I}_{\mathcal{P}}}(\mu, \mu') = \begin{cases} 1 & \text{if } \mu' = \mu i, \\ 0 & \text{otherwise,} \end{cases}$

and the auxiliary concept names V_i', W_i', M' are interpreted in the unique way that satisfies the rest of the axioms.

It is easy to see that $\mathcal{I}_{\mathcal{P}}$ is a model of $\mathcal{O}_{\mathcal{P}}$. Moreover, since every $\mu \in \{1, \ldots, n\}^*$ has exactly n successors with degree greater than 0, it follows that $\mathcal{I}_{\mathcal{P}}$ must be also witnessed: the infima and suprema have to be computed over a finite domain, and hence become minima and maxima. More interesting is that $\mathcal{I}_{\mathcal{P}}$ is indeed a canonical description of all witnessed models:

Lemma 3.9. *Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_{\mathcal{P}}$. There exists a function $f : \Delta^{\mathcal{I}_{\mathcal{P}}} \to \Delta^{\mathcal{I}}$ such that, for every $\mu \in \Delta^{\mathcal{I}_{\mathcal{P}}}$, $C^{\mathcal{I}_{\mathcal{P}}}(\mu) = C^{\mathcal{I}}(f(\mu))$ holds for every concept name C appearing in $\mathcal{O}_{\mathcal{P}}$ and $r_i^{\mathcal{I}}(f(\mu), f(\mu i)) = 1$ holds for every $i, 1 \leq i \leq n$.*

Proof. First recall that the axioms in $\mathcal{T}_{\mathcal{P}}^0$ ensure that V_i, W_i are interpreted as constants in every model, and hence for these concept names the result trivially holds. We now extend it to all other concept names.

The function f is built inductively on the length of μ.

First, since $\mathcal{I}$ is a model of $\mathcal{A}_{\mathcal{P}}^0 \cup \mathcal{O}_0$, there must be a $\delta \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} = \delta$. Since the assertions in $\mathcal{A}_{\mathcal{P}}^0 \cup \mathcal{O}_0$ fixes the interpretation

of all remaining concept names, $f(\varepsilon) = \delta$ satisfies the condition of the lemma.

Let now $\mu \in \Delta^{\mathcal{I}_{\mathcal{P}}}$ be such that $f(\mu)$ has been already defined. By induction, we can assume that $V^{\mathcal{I}}(f(\mu)) = 0.\overleftarrow{v_\mu}$, $W^{\mathcal{I}}(f(\mu)) = 0.\overleftarrow{w_\mu}$, and $M^{\mathcal{I}}(f(\mu)) = 1 - (s+1)^{-(k|\mu|+1)}$ hold. Since $\mathcal{I}$ is a witnessed model of $\langle \top \sqsubseteq \exists r_i.\top, 1 \rangle$ for all $i, 1 \le i \le n$, there exists a $\gamma_i \in \Delta^{\mathcal{I}}$ with $r_i^{\mathcal{I}}(f(\mu), \gamma_i) = 1$.

As $\mathcal{I}$ satisfies all the axioms from $\mathcal{T}_{\mathcal{P}}^i$, it follows, as explained above, that $V^{\mathcal{I}}(\gamma_i) = 0.\overleftarrow{v_\mu v_i} = V^{\mathcal{I}_{\mathcal{P}}}(\mu i)$ and $W^{\mathcal{I}}(\gamma_i) = W^{\mathcal{I}_{\mathcal{P}}}(\mu i)$. Moreover, since it also satisfies the axioms in $\mathcal{O}_0$, we have that $M'^{\mathcal{I}}(\delta) = (s+1)^{-(k|\mu|+1+k)} = (s+1)^{-(k|\mu i|+1)}$. The last two axioms ensure then that $M^{\mathcal{I}}(\gamma_i) = M'^{\mathcal{I}}(\delta) = (s+1)^{-(k|\mu i|+1)}$. Thus, setting $f(\mu i) = \gamma_i$ satisfies the condition of the lemma. $\square$

This means that the search tree, together with a bound for the differences between the words v_ν and w_ν is encoded in every witnessed model of $\mathcal{O}_{\mathcal{P}}$. We now restrict these models to ensure that $v_\nu \ne w_\nu$ for all $\nu \in \{1, \ldots, n\}^*$.

Consider the ontology

$\mathcal{O}'_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{\ne}$

$\mathcal{O}_{\ne} = \{\langle \top \sqsubseteq \forall r_i.(\neg((\neg V \sqcup W) \sqcap (\neg W \sqcup V)) \sqcup M), 1 \rangle \mid 1 \le i \le n\}.$

Recall that in the Łukasiewicz t-norm, $\neg x \oplus y \equiv x \Rightarrow y$. Thus, the above axioms say that for every $i, 1 \le i \le n$ and every $\nu i \in \{1, \ldots, n\}^+$ it holds that

$$((\neg V \sqcup W) \sqcap (\neg W \sqcup V))^{\mathcal{I}}(f(\nu i)) \Rightarrow M^{\mathcal{I}}(f(\nu i)) \ge 1.$$

We know that if $v_{\nu i} = w_{\nu i}$, then the left-hand-side of this implication is 1; since $M^{\mathcal{I}}(f(\nu i)) < 1$, that would violate the restriction. Moreover, we know by the way M was defined that if $v_{\nu i} \ne w_{\nu i}$, then the restriction is satisfied. We thus have reduced the PCP to ontology (in)consistency of L-$\mathcal{ALC}$.

Theorem 3.10. *$\mathcal{P}$ has a solution iff $\mathcal{O}'_{\mathcal{P}}$ is inconsistent.*

Proof. Suppose first that $\mathcal{P}$ has a solution $\nu i \in \{1, \ldots, n\}^+$. We need to show that $\mathcal{O}'_{\mathcal{P}}$ has no model. Suppose that we have a model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, then we will show that $\mathcal{I}$ violates at least one of the axioms from $\mathcal{O}_{\ne}$.

Since $\mathcal{I}$ is a model of $\mathcal{O}_{\mathcal{P}}$, from Lemma 3.9 we know that there is a function $f : \Delta^{\mathcal{I}_{\mathcal{P}}} \to \Delta^{\mathcal{I}}$ such that

- $V^{\mathcal{I}}(f(\nu i)) = V^{\mathcal{I}_{\mathcal{P}}}(\nu i) = 0.\overleftarrow{v_{\nu i}}$,

- $W^{\mathcal{I}}(f(\nu i)) = W^{\mathcal{I}_{\mathcal{P}}}(\nu i) = 0.\overleftarrow{w_{\nu i}}$,

- $M^{\mathcal{I}}(f(\nu i)) = M^{\mathcal{I}_{\mathcal{P}}}(\nu i) = 1 - (s+1)^{k|\nu i|+1}$, and

- $r_i^{\mathcal{I}}(f(\nu), f(\nu i)) = 1.$

Sinc νi is a solution of $\mathcal{P}$, we have that $v_{\nu i} = w_{\nu i}$ and hence

$$(\neg V \sqcup W)^{\mathcal{I}}(f(\nu i)) = (\neg W \sqcup V)^{\mathcal{I}}(f(\nu i)) = 1 - 0.\overleftarrow{v_{\nu i}} + 0.\overleftarrow{v_{\nu i}} = 1.$$

This implies that

$$(\neg((\neg V \sqcup W) \sqcap (\neg W \sqcup V)) \sqcup M)^{\mathcal{I}}(f(\nu i)) = M^{\mathcal{I}}(f(\nu i)) < 1.$$

In particular, this means that

$$\forall r_i.(\neg((\neg V \sqcup W) \sqcap (\neg W \sqcup V)) \sqcup M)^{\mathcal{I}}(f(\nu))$$
$$\le r_i^{\mathcal{I}}(f(\nu), f(\nu i)) \Rightarrow M^{\mathcal{I}}(f(\nu i)) < 1.$$

violating the axiom in $\mathcal{O}_{\ne}$.

For the converse, assume now that $\mathcal{O}'_{\mathcal{P}}$ is inconsistent; we will show that $\mathcal{P}$ has a solution. Since $\mathcal{O}'_{\mathcal{P}}$ is inconsistent, it has no model. In particular, $\mathcal{I}_{\mathcal{P}}$ is not a model of $\mathcal{O}'_{\mathcal{P}}$. But we know that $\mathcal{I}_{\mathcal{P}}$ is a model of $\mathcal{O}_{\mathcal{P}}$, hence $\mathcal{I}_{\mathcal{P}}$ must violate some axiom from $\mathcal{O}_{\ne}$. That is, there must exist a $\nu \in \{1, \ldots, n\}^*$ and $i, 1 \le i \le n$ such that

$$\forall r_i.(\neg((\neg V \sqcup W) \sqcap (\neg W \sqcup V)) \sqcup M)^{\mathcal{I}_{\mathcal{P}}}(\nu) < 1.$$

We will show now that νi is a solution of $\mathcal{P}$.

Since ν has exactly one r_i successor (namely νi) and $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu i) = 1$, it follows that

$$(\neg((\neg V \sqcup W) \sqcap (\neg W \sqcup V)) \sqcup M)^{\mathcal{I}_{\mathcal{P}}}(\nu i) < 1. \tag{3.4}$$

If $\overleftarrow{v_{\nu i}} < \overleftarrow{w_{\nu i}}$, then $(\neg V \sqcup W)^{\mathcal{I}_{\mathcal{P}}}(\nu i) = 1$ holds and as explained before, $(\neg W \sqcup V)^{\mathcal{I}_{\mathcal{P}}}(\nu i) \le M^{\mathcal{I}_{\mathcal{P}}}(\nu i)$ but then

$$(\neg((\neg V \sqcup W) \sqcap (\neg W \sqcup V))^{\mathcal{I}_{\mathcal{P}}}(\nu i) \ge 1 - M^{\mathcal{I}_{\mathcal{P}}}(\nu i),$$

which violates the inequality (3.4). An analogous argument shows that if $\overleftarrow{w_{\nu i}} < \overleftarrow{v_{\nu i}}$ we also obtain a contradiction.

We thus conclude that $\overleftarrow{w_{\nu i}} = \overleftarrow{v_{\nu i}}$ and hence νi is a solution for $\mathcal{P}$. $\quad\square$

Since the PCP is known to be undecidable, we get undecidability of ontology consistency in Ł-$\mathcal{ALC}$.

Corollary 3.11. *Ontology consistency in* Ł-$\mathcal{ALC}$ *is undecidable.*

It should be noted that Ł-$\mathcal{ALC}$ is not the only fuzzy DL that is known to be undecidable. In fact, several undecidability results are known for variants of $\mathcal{ALC}$, using different constructors or axioms, and for a wide variety of semantics, including the product t-norm.

Given these negative results—if we cannot effectively reason in these logics, then they are not suitable for knowledge representation—we turn our attention to a restricted semantics where only finitely many truth degrees are allowed.

4 Finite Lattice Semantics

We have seen that under Gödel semantics, reasoning is not harder than in the crisp case. However, if we use any other continuous t-norm, then it is conjectured that reasoning already becomes undecidable (as we showed for the Łukasiewicz in the previous chapter). One of the main reasons for this huge jump in the complexity of reasoning is that the Gödel t-norm is, in some way, only finitely-valued. Indeed, as we saw in Chapter 2, although the semantics is based on the whole interval $[0, 1]$, we can restrict reasoning to models that only use the truth degrees explicitly appearing in the axioms of the ontology, and forget about all other truth degrees. This property does not hold in any other continuous t-norm: it is always possible to build an ontology whose models require infinitely many different truth degrees. [Exercise?]

Thus, to regain decidability (and a lower complexity) of reasoning, it makes sense to restrict the semantics to finitely many truth degrees. This has the additional advantage of removing the restriction on having a *total order* within these degrees. We will generalize to so-called *residuated lattices*.

4.1 Residuated Lattices

A *lattice* is an algebraic structure $(L, \vee, \wedge)$ over a *carrier set* L with the two binary operations $\vee$ (called the *join*) and $\wedge$ (the *meet*) that satisfy the following properties: they are idempotent, associative, and commutative and satisfy the absorption laws

$$\ell_1 \vee (\ell_1 \wedge \ell_2) = \ell_1 = \ell_1 \wedge (\ell_1 \vee \ell_2)$$

for every $\ell_1, \ell_2 \in L$.

In every lattice L we have a partial order $\leq$, defined for every $\ell_1, \ell_2 \in L$ by $\ell_1 \leq \ell_2$ iff $\ell_1 \wedge \ell_2 = \ell_1$. An *antichain* of L is a subset $T \subseteq L$

whose elements are pairwise incomparable; that is, for every $\ell_1, \ell_2 \in T$, if $\ell_1 \leq \ell_2$, then $\ell_1 = \ell_2$.

We say that the lattice L is *distributive* if $\vee$ and $\wedge$ distribute over each other:

$$(\ell_1 \wedge \ell_2) \vee \ell_3 = (\ell_1 \vee \ell_3) \wedge (\ell_2 \vee \ell_3)$$
$$(\ell_1 \vee \ell_2) \wedge \ell_3 = (\ell_1 \wedge \ell_3) \vee (\ell_2 \wedge \ell_3),$$

and is *bounded* if it has a *minimum* and a *maximum* element, denoted as $\mathbf{0}$ and $\mathbf{1}$, respectively. It is *complete* if joins and meets of arbitrary subsets $T \subseteq L$, denoted by $\bigvee_{t \in T} t$ and $\bigwedge_{t \in T} t$ respectively, exist.

Clearly, every finite lattice is also complete, and every complete lattice is bounded, with $\mathbf{0} = \bigwedge_{t \in L} t$ and $\mathbf{1} = \bigvee_{t \in L} t$.

A *De Morgan lattice* is a bounded distributive lattice extended with a unary operation $\sim$, called the *(De Morgan) negation* which is: involutive ($\sim \sim \ell = \ell$), antimonotonic (if $\ell_1 \leq \ell_2$, then $\sim \ell_2 \leq \sim \ell_1$), and satisfies the De Morgan laws $\sim(\ell_1 \vee \ell_2) = \sim \ell_1 \wedge \sim \ell_2$ and $\sim(\ell_1 \wedge \ell_2) = \sim \ell_1 \vee \sim \ell_2$ for every $\ell_1, \ell_2 \in L$.

Definition 4.1 (Residuated lattice)**.** A *residuated lattice* is a lattice $(L, \vee, \wedge)$ extended with two binary operators $\otimes$ and $\Rightarrow$ satisfying the following properties:

- $\otimes$ is associative, commutative, monotonic and has $\mathbf{1}$ as its unit

- $\ell_1 \otimes \ell_2 \leq \ell_3$ iff $\ell_2 \leq \ell_1 \Rightarrow \ell_3$ holds for every $\ell_1, \ell_2, \ell_3 \in L$.

$\triangle$

In this case, we have $\ell_1 \Rightarrow \ell_2 = \bigvee\{\ell \mid \ell_1 \otimes \ell \leq \ell_2\}$ for all $\ell_1, \ell_2 \in L$.

A simple example of a residuated lattice is a distributive lattice, with $\otimes$ given by the infimum $\wedge$ and $\ell_1 \Rightarrow \ell_2 = \bigvee\{\ell \mid \ell_1 \wedge \ell \leq \ell_2\}$. [Exercise: show that this is a residuated lattice; what happens if the lattice is not distributive?]

For consistency, we will call the operator $\otimes$ a t-norm, and $\Rightarrow$ its residuum. If $\otimes = \wedge$, then we will call this the Gödel t-norm.

As it was the case for t-norms from the previous chapters, we have the following simple properties: for every $\ell_1, \ell_2 \in L$,

- $\mathbf{1} \Rightarrow \ell_1 = \ell_1$, and

- $\ell_1 \leq \ell_2$ iff $\ell_1 \Rightarrow \ell_2 = \mathbf{1}$.

If L is a residuated De Morgan lattice then the *t-conorm* $\oplus$ is given by $\ell_1 \oplus \ell_2 := \sim(\sim \ell_1 \otimes \sim \ell_2)$. The t-conorm of the Gödel t-norm is the supremum operator $\vee$.

The *precomplement* $\ominus$ is given by $\ominus \ell = \ell \Rightarrow \mathbf{0}$. For the Gödel t-norm, the precomplement defines the Gödel negation

$$\ominus \ell = \begin{cases} \mathbf{0} & \text{if } \ell \neq \mathbf{0}, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

For the following, L is *always* a finite residuated De Morgan lattice.

For every such L, we define the logic $\mathcal{ALC}_L$, that has the same syntax as crisp $\mathcal{ALC}$. We will only focus on the problem of *concept satisfiability*, and hence do not introduce any ABox axioms. The *axioms* of this logic are GCIs of the form $\langle C \sqsubseteq D, \ell \rangle$, where C, D are $\mathcal{ALC}_L$ concepts and $\ell \in L$.

Definition 4.2 (semantics of $\mathcal{ALC}_L$)**.** A *fuzzy interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called the *domain* and $\cdot^{\mathcal{I}}$ is a function mapping:

- every concept name $A \in \mathsf{N_C}$ to a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to L$, and

- every role name $r \in \mathsf{N_R}$ to a function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to L$.

This function is extended to concept descriptions as follows. For every $x \in \Delta^{\mathcal{I}}$:

- $\top^{\mathcal{I}}(x) = \mathbf{1}$, $\bot^{\mathcal{I}}(x) = \mathbf{0}$,

- $(\neg C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$,

- $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$,

- $(C \sqcup D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$,

- $(\exists r.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)\}$,

- $(\forall r.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)\}$.

$\mathcal{I}$ *satisfies* the GCI $\langle C \sqsubseteq D, \ell \rangle$ if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell$ for every $x \in \Delta^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$. $\triangle$

As said before, we are interested in the problem of satisfiability of a concept, which intuitively describes how true a concept can be in a model of an ontology.

Definition 4.3 (satisfiability). A concept C is ℓ-satisfiable w.r.t. an ontology $\mathcal{O}$, for $\ell \in L$, if there exists a model $\mathcal{I}$ of $\mathcal{O}$ and $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(x) \geq \ell$. $\triangle$

[Notice that satisfiability is independent of any assertional axioms; thus, in the following, whenever we speak of an ontology, we will w.l.o.g. assume that it is only a TBox.]

We will show that satisfiability is decidable in exponential time using a reduction to emptiness of automata.

4.2 Looping Tree Automata

We use a very simple kind of automata over infinite k-ary trees, for some fixed $k \in \mathbb{N}$, that has no alphabet, and no additional acceptance condition.

We will represent the *nodes* of an infinite k-ary tree by words from $\{1, \ldots, k\}^*$, where ε is the root node and for every $u \in \{1, \ldots, k\}^*$, ui is the i-th successor of the node u. [Binary Tree Drawing] A *path* is a sequence $u_1, \ldots, u_m$ of nodes such that $u_1 = \varepsilon$ and u_{i+1} is a successor of u_i for every $i \geq 1$.

For brevity, we denote as K the set $\{1, \ldots, k\}$.

Definition 4.4 (looping automata). A *looping tree automaton* (LA) over k-ary trees is a tuple $\mathcal{A} = (Q, I, \Delta)$ where

- Q is a finite set of *states*,
- $I \subseteq Q$ is a set of *initial states*, and
- $\Delta \subseteq Q^{k+1}$ is the *transition relation*.

A *run* of $\mathcal{A}$ is a function $r : K^* \to Q$ that assigns states to every node of the tree K^*, such that:

- $r(\varepsilon) \in I$, and
- $(r(u), r(u1), \ldots, r(uk)) \in \Delta$ for every $u \in K^*$.

$\triangle$

The relevant decision problem for LA is to decide whether there is a run of a given automaton $\mathcal{A}$ or not. This is known as the *emptiness problem*.

Let $\mathcal{A}$ be an LA. The emptiness problem of $\mathcal{A}$ can be decided in polynomial time using a "bottom-up" procedure that detects all states that cannot appear in a run ("bad" states) and then verifies that there is at least one initial state that is not bad (and which will be the root of the existing run).

The set of bad states is built iteratively as follows:

1. $\mathsf{bad}_0 = \emptyset$

2. for every $i \geq 0$, let S_i be the set of all states $q \in Q$ such that, for every transition $(q, q_1, \ldots, q_k) \in \Delta$, we have $\{q_1, \ldots, q_k\} \in \mathsf{bad}_i$. Then $\mathsf{bad}_{i+1} = \mathsf{bad}_i \cup S_i$.

It can be shown that $\mathcal{A}$ has a run iff $I \setminus \mathsf{bad}_{|Q|} \neq \emptyset$.

Thus, to test emptiness of $\mathcal{A}$, we need to iterate $|Q|$ times the computation of bad states. Each of these iterations needs to look at every transition in Δ exactly once, and do a set comparison with the result of the previous iteration. Thus, in total this algorithm takes $\mathcal{O}(Q^{k+2})$ steps; that is, it runs in time polynomial on the size of $\mathcal{A}$.

4.3 Deciding Satisfiability

To decide ℓ-satisfiability of a concept C w.r.t. an ontology $\mathcal{O}$, we will build an automaton $\mathcal{A}_{C,\mathcal{O}}$ whose runs represent tree-like models of $\mathcal{O}$ with domain K^*. The states of this automaton will represent the membership degree of a node to all relevant concepts. Then, ℓ-satisfiability is ensured by restricting the root node to belong to the concept C with a degree at least ℓ.

This construction is based on the fact that an ontology has a witnessed model if and only if it has a well-structured tree-shaped model,

called a *Hintikka tree*. To define these models, we need the notion of subconcepts.

Definition 4.5 (subconcepts)**.** The set $\mathsf{sub}(C)$ of *subconcepts* of a concept C is inductively defined as follows:

- $\mathsf{sub}(A) = A$ for $A \in \mathsf{N_C} \cup \{\top, \bot\}$,

- $\mathsf{sub}(C \sqcap D) = \{C \sqcap D\} \cup \mathsf{sub}(C) \cup \mathsf{sub}(D)$,

- $\mathsf{sub}(C \sqcup D) = \{C \sqcup D\} \cup \mathsf{sub}(C) \cup \mathsf{sub}(D)$,

- $\mathsf{sub}(\neg C) = \{\neg C\} \cup \mathsf{sub}(C)$,

- $\mathsf{sub}(\exists r.C) = \{\exists r.C\} \cup \mathsf{sub}(C)$, and

- $\mathsf{sub}(\forall r.C) = \{\forall r.C\} \cup \mathsf{sub}(C)$.

For a concept C and an ontology $\mathcal{O}$,

$$\mathsf{sub}(C, \mathcal{O}) := \mathsf{sub}(C) \cup \bigcup_{\langle D \sqsubseteq E, \ell \rangle \in \mathcal{O}} \mathsf{sub}(D) \cup \mathsf{sub}(E).$$

The nodes of Hintikka trees are labeled with Hintikka functions: fuzzy sets over $\mathsf{sub}(C, \mathcal{O}) \cup \{\rho\}$, where ρ is an arbitrary new element, that are propositionally consistent.

Definition 4.6 (Hintikka function)**.** A Hintikka function for $C, \mathcal{O}$ is a function $H : \mathsf{sub}(C, \mathcal{O}) \cup \{\rho\} \to L$ such that

- $H(\top) = \mathbf{1}, H(\bot) = \mathbf{0}$,

- $H(D \sqcap E) = H(D) \otimes H(E)$,

- $H(D \sqcup E) = H(D) \oplus H(E)$, and

- $H(\neg C) = \sim H(C)$.

H is *compatible* with the GCI $\langle D \sqsubseteq E, \ell \rangle \in \mathcal{O}$ if $H(D) \Rightarrow H(E) \geq \ell$. $\triangle$

The arity of the Hintikka trees is determined by the number of existential and universal restrictions in $\mathsf{sub}(C, \mathcal{O})$. Intuitively, each successor will be the witness of one of these restrictions. For the construction it will be important to know which successor witnesses which restriction; thus we fix an arbitrary bijection

$$\varphi : \{E \in \mathsf{sub}(C, \mathcal{O}) \mid E \text{ is of the form } \exists r.F \text{ or } \forall r.F\} \to K.$$

Finally, a local condition between nodes and their successors ensures that the existential and universal restrictions are satisfied.

Definition 4.7 (Hintikka condition)**.** A tuple $(H_0, H_1, \ldots, H_k)$ of Hintikka functions for $C, \mathcal{O}$ satisfies the *Hintikka condition* if:

1. For every existential restriction $\exists r.D \in \mathsf{sub}(C, \mathcal{O})$, it holds that $H_0(\exists r.D) = H_{\varphi(\exists r.D)}(\rho) \otimes H_{\varphi(\exists r.D)}(D)$. Additionally, for every quantified concept F of the form $\exists r.E$ or $\forall r.E$ it holds that $H_0(\exists r.D) \geq H_{\varphi(F)}(\rho) \otimes H_{\varphi(F)}(D)$.

2. For every value restriction $\forall r.D \in \mathsf{sub}(C, \mathcal{O})$, it holds that $H_0(\forall r.D) = H_{\varphi(\forall r.D)}(\rho) \Rightarrow H_{\varphi(\forall r.D)}(D)$. Additionally, for every quantified concept F of the form $\exists r.E$ or $\forall r.E$ it holds that $H_0(\forall r.D) \leq H_{\varphi(F)}(\rho) \Rightarrow H_{\varphi(F)}(D)$.

Each of these conditions ensures two things: first, that a witness of each quantified formula exists, and second, that the semantics of the quantifiers are satisfied.

Definition 4.8 (Hintikka tree)**.** A *Hintikka tree* for $C, \mathcal{O}$ is a labeled k-ary tree $\mathbf{T}$ such that, for every node $u \in K^*$, $\mathbf{T}(u)$ is a Hintikka function compatible with $\mathcal{O}$, and $(\mathbf{T}(u), \mathbf{T}(u1), \ldots, \mathbf{T}(uk))$ satisfies the Hintikka condition.

Given a Hintikka tree $\mathbf{T}$ for $C, \mathcal{O}$, we define the interpretation $\mathcal{I}_{\mathbf{T}} = (K^*, \cdot^{\mathcal{I}_{\mathbf{T}}})$, where $A^{\mathcal{I}_{\mathbf{T}}}(u) = \mathbf{T}(u)(A)$ for every $A \in \mathsf{N_C}$, and

$$r^{\mathcal{I}_{\mathbf{T}}}(u, v) = \begin{cases} \mathbf{T}(v)(\rho) & \text{if } v = ui, 1 \leq i \leq k, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Lemma 4.9. *Let* $\mathbf{T}$ *be a Hintikka tree for* $C, \mathcal{O}$*. For every* $D \in$ sub$(C, \mathcal{O})$ *and* $u \in K^*$ *it holds that* $D^{\mathcal{I}_\mathbf{T}}(u) = \mathbf{T}(u)(D)$.

Proof. The proof is by induction on the structure of D. If D is a concept name, the result follows trivially from the definition of $\mathcal{I}_\mathbf{T}$. Since $\mathbf{T}(u)$ is a Hintikka function for every $u \in K^*$, it follows that:

- $\top^{\mathcal{I}_\mathbf{T}}(u) = \mathbf{1} = \mathbf{T}(u)(\top)$ and $\perp^{\mathcal{I}_\mathbf{T}}(u) = \mathbf{0} = \mathbf{T}(u)(\perp)$,

- $(D_1 \sqcap D_2)^{\mathcal{I}_\mathbf{T}}(u) = D_1^{\mathcal{I}_\mathbf{T}}(u) \otimes D_2^{\mathcal{I}_\mathbf{T}}(u) = \mathbf{T}(u)(D_1) \otimes \mathbf{T}(u)(D_2) = \mathbf{T}(u)(D_1 \sqcap D_2)$, and analogously for disjunction,

- $(\neg D)^{\mathcal{I}_\mathbf{T}}(u) = \sim D^{\mathcal{I}_\mathbf{T}}(u) = \sim \mathbf{T}(u)(D) = \mathbf{T}(u)(\neg D)$.

For the quantified concepts, we have:

$$(\exists r.D)^{\mathcal{I}_\mathbf{T}}(u) = \bigvee_{v \in K^*} r^{\mathcal{I}_\mathbf{T}}(u, v) \otimes D^{\mathcal{I}_\mathbf{T}}(v) = \bigvee_{i=1}^{k} r^{\mathcal{I}_\mathbf{T}}(u, ui) \otimes D^{\mathcal{I}_\mathbf{T}}(ui)$$
$$= r^{\mathcal{I}_\mathbf{T}}(u, u\varphi(\exists r.D)) \otimes D^{\mathcal{I}_\mathbf{T}}(u\varphi(\exists r.D))$$
$$= \mathbf{T}(u\varphi(\exists r.D))(\rho) \otimes \mathbf{T}(u\varphi(\exists r.D))(D)$$
$$= \mathbf{T}(u)(\exists r.D).$$

and analogously for the value restrictions. $\qquad \square$

In particular, Hintikka trees are models of the ontology.

Corollary 4.10. *If* $\mathbf{T}$ *is a Hintikka tree for* $C, \mathcal{O}$*, then* $\mathcal{I}_\mathbf{T}$ *is a model of* $\mathcal{O}$.

Proof. Let $\langle C \sqsubseteq D, \ell \rangle \in \mathcal{O}$ and $u \in K^*$. Since $\mathbf{T}(u)$ is compatible with this axiom, it follows that

$$C^{\mathcal{I}_\mathbf{T}}(u) \Rightarrow C^{\mathcal{I}_\mathbf{T}}(u) = \mathbf{T}(u)(C) \Rightarrow \mathbf{T}(u)(D) \geq \ell.$$

$\qquad \square$

This means that we can decide satisfiability by looking for Hintikka trees.

Theorem 4.11. *Let* C *be an* $\mathcal{ALC}_L$ *concept,* $\mathcal{O}$ *an ontology, and* $\ell \in L$. C *is* ℓ-*satisfiable w.r.t.* $\mathcal{O}$ *(in a witnessed model) iff there is a Hintikka tree* $\mathbf{T}$ *for* $C, \mathcal{O}$ *such that* $\mathbf{T}(\varepsilon)(C) \geq \ell$.

Proof. The *if* direction was shown in Corollary 4.10.

For the *only if* direction, let $\mathcal{I}$ be a witnessed model of $\mathcal{O}$ and $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(x) \geq \ell$. We build inductively a function $f : K^* \to \Delta^{\mathcal{I}}$ and a Hintikka tree $\mathbf{T}$ such that $\mathbf{T}(u)(D) = D^{\mathcal{I}}(f(u))$ holds for every $u \in K^*$ and $D \in$ sub$(C, \mathcal{O})$.

First we set $f(\varepsilon) = x$ and for every $D \in$ sub$(C, \mathcal{O})$, $\mathbf{T}(\varepsilon)(D) = D^{\mathcal{I}}(x)$. Since $\mathcal{I}$ is a model, it directly follows that $\mathbf{T}(\varepsilon)$ is a Hintikka function compatible with $\mathcal{O}$. Moreover, we have that $\mathbf{T}(\varepsilon)(C) = C^{\mathcal{I}}(x) \geq \ell$.

Let now $u \in K^*$ be a node for which $f(u)$ and $\mathbf{T}(u)$ have been already defined. Given an existential restriction $\exists r.D \in$ sub$(C, \mathcal{O})$, since $\mathcal{I}$ is witnessed, we know that there exists a $y_{\exists r.D} \in \Delta^{\mathcal{I}}$ such that $(\exists r.D)^{\mathcal{I}}(f(u)) = r^{\mathcal{I}}(f(u), y_{\exists r.D}) \otimes D^{\mathcal{I}}(y_{\exists r.D})$. Moreover, for every $z \in \Delta^{\mathcal{I}}$, $(\exists r.D)^{\mathcal{I}}(f(u)) \geq r^{\mathcal{I}}(f(u), z) \otimes D^{\mathcal{I}}(z)$. Dually, for every value restriction $\forall r.D \in$ sub$(C, \mathcal{O})$ there is a $y_{\forall r.D} \in \Delta^{\mathcal{I}}$ with $(\forall r.D)^{\mathcal{I}}(f(u)) = r^{\mathcal{I}}(f(u), y_{\forall r.D}) \Rightarrow D^{\mathcal{I}}(y_{\forall r.D})$ and $(\forall r.D)^{\mathcal{I}}(f(u)) \leq r^{\mathcal{I}}(f(u), z) \Rightarrow D^{\mathcal{I}}(z)$.

For every $i, 1 \leq i \leq k$, we define $f(ui) = y_{\varphi^{-1}(i)}$. From the above considerations, it follows that $(\mathbf{T}(u), \mathbf{T}(u1), \ldots, \mathbf{T}(uk))$ satisfies the Hintikka condition.

Thus $\mathbf{T}$ is a Hintikka tree for $C, \mathcal{O}$ with $\mathbf{T}(\varepsilon)(C) \geq \ell$. $\qquad \square$

To decide the existence of such a Hintikka tree, we use a looping automaton over k-ary trees. The idea is that the runs of this automaton correspond exactly to Hintikka trees satisfying the root condition. Then, concept satisfiability is reduced to the emptiness problem of looping automata.

Definition 4.12 (Hintikka automaton). Let C be an $\mathcal{ALC}_L$ concept, $\mathcal{O}$ an ontology and $\ell \in L$. The *Hintikka automaton* for $C, \mathcal{O}, \ell$ is the LA $\mathcal{A}_{C,\mathcal{O},\ell} = (Q, I, \Delta)$ where:

- Q is the set of all Hintikka functions for $C, \mathcal{O}$ compatible with $\mathcal{O}$,

- I contains all Hintikka function H such that $H(C) \geq \ell$, and

- Δ is the set of all $(k+1)$-tuples of Hintikka functions that satisfy the Hintikka condition.

It is easy to see that $\mathcal{A}_{C,\mathcal{O},\ell}$ has a run if and only if there is a Hintikka tree as in Theorem 4.11. We thus obtain the following result.

Corollary 4.13. *A concept C is ℓ-satisfiable w.r.t. an ontology $\mathcal{O}$ (in a witnessed model) iff the automaton $\mathcal{A}_{C,\mathcal{O},\ell}$ has a run.*

We can decide whether an automaton $\mathcal{A}$ has a run in polynomial time on the number of states of $\mathcal{A}$. Corollary 4.13 shows then that concept satisfiability is decidable, and needs only time polynomial on the number of states of $\mathcal{A}_{C,\mathcal{O},\ell}$.

Since states are functions $H : \mathsf{sub}(C,\mathcal{O}) \to L$, the number of states is bounded by the number of such functions; thus, there are at most $|L|^{|\mathsf{sub}(C,\mathcal{O})|}$ states in $\mathcal{A}_{C,\mathcal{O},\ell}$. Notice that $|\mathsf{sub}(C,\mathcal{O})|$ is bounded polynomially on the size of the input $C,\mathcal{O}$, and L is a constant since we assume that the lattice is fixed for setting the semantics.

Thus, the number of states of the automaton is exponential on the size of $C,\mathcal{O}$. In total, this means that satisfiability in $\mathcal{ALC}_L$ can be decided in exponential time on the size of $C,\mathcal{O}$.

Theorem 4.14. *Satisfiability of $\mathcal{ALC}_L$ concepts (w.r.t. witnessed models) is* EXPTIME-*complete.*