Fuzzy Description Logics

Exercise Sheet 7

Dr. Rafael Peñaloza / Dr. Felix Distel
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Exercise 1

Assume that we have selected x_{i+1} such that $r^{\mathcal{I}}(x_i, x_{i+1}) \geq 1 - 0.2A^{\mathcal{I}}(x_i)$ as described in the tutorial.

We show that then $0.4A^{\mathcal{I}}(x_i) \leq A^{\mathcal{I}}(x_{i+1}) \leq 0.6A^{\mathcal{I}}(x_i)$ holds (This claim is *different from the tutorial*).

Then from $\langle A \sqsubseteq \forall r.(A \sqcup A), 1\rangle$ we obtain

$$\begin{aligned}
A^{\mathcal{I}}(x_i) &\leq \overbrace{r^{\mathcal{I}}(x_i, x_{i+1})}^{>0.9} \Rightarrow \overbrace{2A^{\mathcal{I}}(x_{i+1})}^{\leq 0.9} \\
&= 1 - r^{\mathcal{I}}(x_i, x_{i+1}) + 2A^{\mathcal{I}}(x_{i+1}) \\
&\leq 0.2A^{\mathcal{I}}(x_i) + 2A^{\mathcal{I}}(x_{i+1}).
\end{aligned}$$

Thus $A^{\mathcal{I}}(x_i) - 0.2A^{\mathcal{I}}(x_i) \leq 2A^{\mathcal{I}}(x_{i+1})$ and therefore $0.4A^{\mathcal{I}}(x_i) \leq A^{\mathcal{I}}(x_{i+1})$.

Likewise, from $\langle \exists r.(A \sqcup A) \sqsubseteq A, 1\rangle$ we obtain

$$\begin{aligned}
A^{\mathcal{I}}(x_i) &\geq \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x_i, y) \otimes (2A^{\mathcal{I}}(y)) \\
&\geq r^{\mathcal{I}}(x_i, x_{i+1}) \otimes (2A^{\mathcal{I}}(x_{i+1})) \\
&= \max\{r^{\mathcal{I}}(x_i, x_{i+1}) + 2A^{\mathcal{I}}(x_{i+1}) - 1, 0\} \\
&\geq r^{\mathcal{I}}(x_i, x_{i+1}) + 2A^{\mathcal{I}}(x_{i+1}) - 1 \\
&\geq 2A^{\mathcal{I}}(x_{i+1}) - 0.2A^{\mathcal{I}}(x_i)
\end{aligned}$$

Thus $A^{\mathcal{I}}(x_i) + 0.2A^{\mathcal{I}}(x_i) \geq 2A^{\mathcal{I}}(x_{i+1})$ and therefore $0.6A^{\mathcal{I}}(x_i) \geq A^{\mathcal{I}}(x_{i+1})$.

We have thus proved that $0.4A^{\mathcal{I}}(x_i) \leq A^{\mathcal{I}}(x_{i+1}) \leq 0.6A^{\mathcal{I}}(x_i)$ holds for all $i \in \mathbb{N}$. With $x_0 = a^i$ and knowing that $A^{\mathcal{I}}(x_0) = 0.4$ we obtain a sequence $(x_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ it holds that $A^{\mathcal{I}}(x_i) > 0$ and $A^{\mathcal{I}}(x_i) > A^{\mathcal{I}}(x_{i+1})$. Hence there must occur infinitely many truth values in $\mathcal{I}$.