Introduction to Automatic Structures

Solution to Exercise 17 b)

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Winter Semester 2011/2012

Exercise 17

b) In order to ensure that $(L, \leq)$ represents a tree we need to ensure that

- L contains a least element (the "root"): $\phi_1 = \exists u.\forall v.u \leq v$
- There is only one "path" to each element; in other words for each $z \in L$ the set of elements of L that are smaller than z is totally ordered:
  $\phi_2 = \forall z.\forall x.\forall y.(x \leq z) \wedge (y \leq z) \rightarrow (x \leq y) \vee (y \leq x)$
- Every interval contains at most a finite number of nodes.
  $\phi_3 = \forall x.\forall y.\neg\exists^\infty z.(x \leq z) \wedge (z \leq y)$.

To ensure infinite outdegree we proceed as follows.

- We can define a predicate for the immediate successor relation:
  $S(x, y) = (x \leq y) \wedge \neg(x = y) \wedge \big(\forall z.(x \leq z) \wedge (z \leq y) \rightarrow (x = z) \vee (y = z)\big)$
- The tree has infinite outdegree if there is a node with an infinite number of immediate successors: $\phi_4 = \exists x.\exists^\infty y.S(x, y)$

The full formula would thus be $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$.