Exercise 17

b) In order to ensure that \((L, \leq)\) represents a tree we need to ensure that

- \(L\) contains a least element (the “root”):
  \[ \phi_1 = \exists u. \forall v. u \leq v \]

- There is only one “path” to each element; in other words for each \(z \in L\) the set of elements of \(L\) that are smaller than \(z\) is totally ordered:
  \[ \phi_2 = \forall z. \forall x. \forall y. (x \leq z) \land (y \leq z) \rightarrow (x \leq y) \lor (y \leq x) \]

- Every interval contains at most a finite number of nodes.
  \[ \phi_3 = \forall x. \forall y. \neg \exists \infty z. (x \leq z) \land (z \leq y). \]

To ensure infinite outdegree we proceed as follows.

- We can define a predicate for the immediate successor relation:
  \[ S(x, y) = (x \leq y) \land \neg (x = y) \land (\forall z. (x \leq z) \land (z \leq y) \rightarrow (x = z) \lor (y = z)) \]

- The tree has infinite outdegree if there is a node with an infinite number of immediate successors:
  \[ \phi_4 = \exists x. \exists \infty y. S(x, y) \]

The full formula would thus be \(\phi = \phi_1 \land \phi_2 \land \phi_3 \land \phi_4\).