Term Rewriting Systems
Exercise Sheet 7
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Winter Semester 2011/2012

Exercise 32
The following problem is known as Hilbert's 10th problem, and it has been proved to be undecidable.

**Given:** One polynomial \( P \in \mathbb{Z}[X_1, \ldots, X_n] \).

**Question:** Are there \( a_1, \ldots, a_n \in \mathbb{N} \) such that \( P(a_1, \ldots, a_n) = 0 \)?

Show that the undecidability of Hilbert's 10th problem implies the undecidability of the following problem, which you know from the lecture:

**Given:** Two polynomials \( P, Q \in \mathbb{N}[X_1, \ldots, X_n] \) and a decidable set \( A \subseteq \mathbb{N} \setminus \{0\} \).

**Question:** Is \( P \succ_A Q \)? That is, is \( P(a_1, \ldots, a_n) > Q(a_1, \ldots, a_n) \) for all \( a_1, \ldots, a_n \in A \)?

Exercise 33
Let \( P \in \mathbb{Z}[X] \) be a polynomial with one indeterminate and coefficients in \( \mathbb{Z} \).

a) Prove for all \( r \in \mathbb{Z} \): if \( P(r) = 0 \), then \( r | a_0 \), i.e. any root of \( P \) divides \( a_0 \).

b) Devise a decision procedure, which for each polynomial \( P \in \mathbb{Z}[X] \) decides whether \( P \) has a root in \( \mathbb{Z} \).

c) Show that for polynomials with more than one indeterminate, the roots need not satisfy such a property.

Exercise 34
Consider the TRS \( R = \{ g(x, g(y, z)) \rightarrow g(g(x, y), z), \ g(g(x, y), z) \rightarrow g(y, y) \} \). Use a polynomial interpretation to prove that \( R \) terminates.

**Hint:** Try \( p_g = xy + y^2 \).

Exercise 35
A TRS \( R \) is called right-irreducible if each \( r \) with \( \ell \rightarrow r \in R \) is irreducible. Prove or refute the following claim: If \( R \) is right-ground and right-irreducible, then \( R \) is terminating.

Exercise 36
Prove Lemma 5.20 from the lecture: Let \( \mathcal{A} \) be a monotone polynomial interpretation of \( \Sigma \). Then \( f^\mathcal{A} \) is a monotone function for each \( f \in \Sigma \).