TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Faculty of Computer Science Chair of Automata Theory

FUZZY LOGIC

Felix Distel

Dresden, WS 2012/13

Course Material

- *Metamathematics of Fuzzy Logic* by Petr Hájek
- available on course website:
  - Slides
  - Lecture Notes (from a previous semester)
  - Exercise Sheets

Contact Information

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- `lat.inf.tu-dresden.de/teaching/ws2012-2012/FL/`

Exams

Oral exams at the end of the semester or during semester break

Classical Logic

- suited for properties that are
  - identifiable,
  - distinct,
  - clear-cut.
- Examples:
  - days of the week,
  - marital status,
  - . . .

Imprecise Knowledge

Is Italy a small country?

Imprecise Knowledge

Is Italy a small country?
Depends.

Imprecise Knowledge

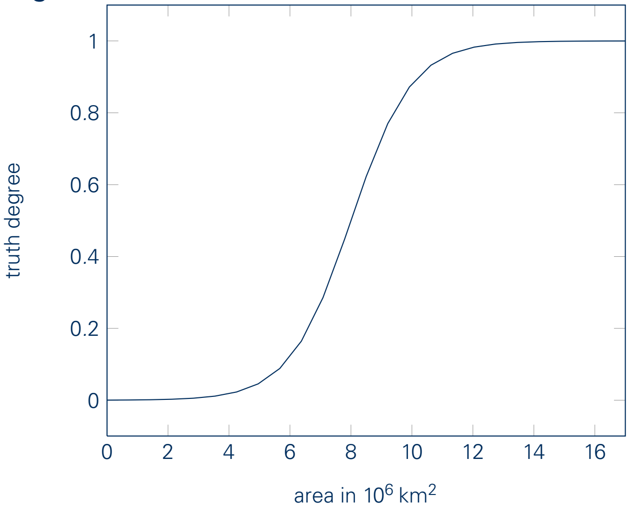
Is Italy a small country?
Depends.

Other examples for fuzzy properties

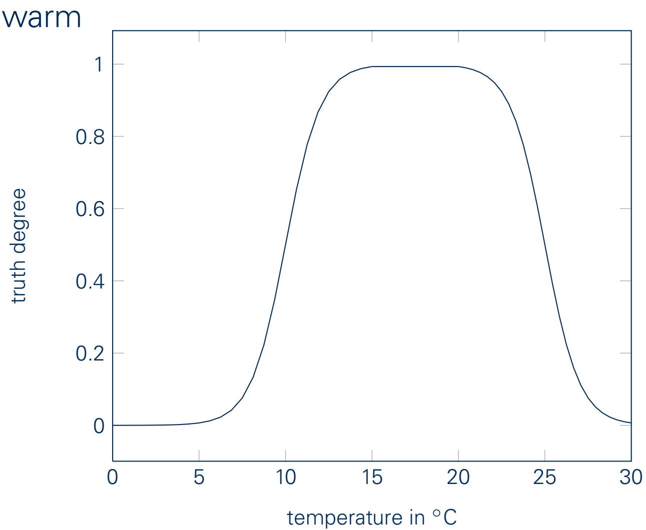
- old
- warm
- tall
- ...

Degrees of Membership

large



Degrees of Membership



Crisp vs. Fuzzy Logics

- Crisp Logics: Only truth values 1 and 0.
  $\Longrightarrow$ characteristic function
- Fuzzy Logics: Truth values from the interval $[0, 1]$.
  $\Longrightarrow$ membership function

Fuzzy vs. Probabilistic Logics

Both use truth values

- Fuzzy Logics: vagueness
  - statement is neither completely true nor false
  - e.g. "The Dresden TV Tower is a tall building."
- Probabilistic Logics: belief or uncertainty
  - statement is either true nor false, but outcome unknown
  - e.g. "Tomorrow it will rain."

Question

How to interpret conjunction?

For the country of **`Turkey`** we might have:

- membership in **`Huge`**: 0.046,
- membership in **`Asian`**: 0.969

What is the membership degree of **`Turkey`** in **`Huge`** $\sqcap$ **`Asian`**?

How to interpret conjunction?

For the country of `Turkey` we might have:

- membership in `Huge`: 0.046,
- membership in `Asian`: 0.969

What is the membership degree of `Turkey` in `Huge` $\sqcap$ `Asian`?

Possible choices

- Minimum of 0.046 and 0.969
- Product of 0.046 and 0.969
- ...

$\Longrightarrow$ There is not just one fuzzy logic!

Generalize Operators

Classical logical operators, such as

- conjunction,
- disjunction,
- negation, and
- implication

need to be generalized.
Generalizations should be

- truth functional
- “behave well” logically (e.g. conjunction should be associative, commutative, etc.)

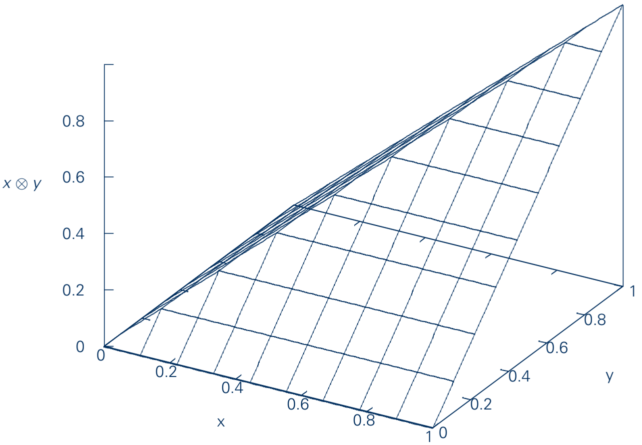
Definition

Binary operator $\otimes\colon [0,1] \times [0,1] \to [0,1]$

- associative,
- commutative,
- monotone, and
- has unit 1.

Fundamental continuous t-norms

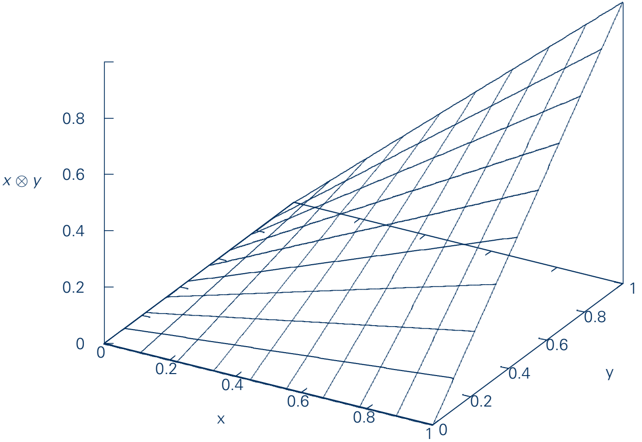
Gödel: $x \otimes y = \min(x, y)$



Fundamental continuous t-norms

Gödel: $x \otimes y = \min(x, y)$

Product: $x \otimes y = x \cdot y$

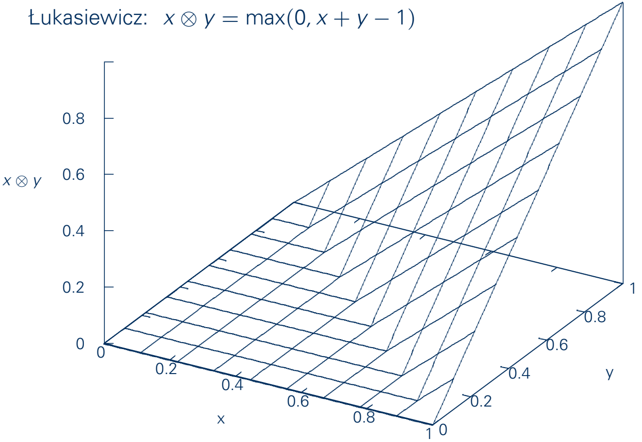


Fundamental continuous t-norms

Gödel: $x \otimes y = \min(x, y)$

Product: $x \otimes y = x \cdot y$

Łukasiewicz: $x \otimes y = \max(0, x + y - 1)$



Truth Functions for Boolean Connectives

Connective	Truth Function	Definition
conjunction (&)	t-norm ($\otimes$)	associative, commutative, monotone, unit 1, (usually also continuous)
implication ($\rightarrow$)	?	
negation ($\neg$)	?	
disjunction ($\vee$)	?	

Modus Ponens in the Crisp Case

$\phi \wedge (\phi \to \psi)$ then ψ.

Fuzzy Generalization of Modus Ponens

$x \otimes \underbrace{(x \Rightarrow y)}_{z} \leq y$

Residuum

Choose z maximal with this property:

$$x \Rightarrow y = \max\{z \mid x \otimes z \leq y\}$$

Uniqueness of Residuum

Lemma 1.2

For every continous t-norm $\otimes$

$$x \Rightarrow y = \max\{z \mid x \otimes z \leq y\}$$

is the unique operator satisfying

$$z \leq x \Rightarrow y \text{ iff } x \otimes z \leq y$$

Truth Functions for Boolean Connectives

Connective	Truth Function	Definition
conjunction (&)	t-norm ($\otimes$)	associative, commutative, monotone, unit 1, (usually also continuous)
implication ($\rightarrow$)	residuum ($\Rightarrow$)	$x \otimes y \leq z$ iff $y \leq x \Rightarrow z$
negation ($\neg$)	precomplement $\ominus$	$x \Rightarrow 0$
disjunction ($\vee$)	?	

Definition

Given (a_i, b_i), $i \in \mathcal{I}$ family disjoint open intervals, $\otimes_i$, $i \in \mathcal{I}$ family of t-norms

$$x \otimes y = \begin{cases} s_i^{-1}\big(s_i(x) \otimes_i s_i(y)\big) & \text{if } x, y \in (a_i, b_i) \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

where

$$s_i(x) = \frac{x - a_i}{b_i - a_i}$$

is the *ordinal sum* $\sum_{i \in \mathcal{I}} (\otimes_i, a_i, b_i)$.

Plots of Ordinal Sums

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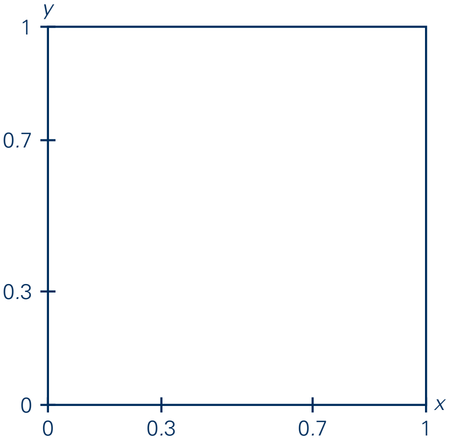

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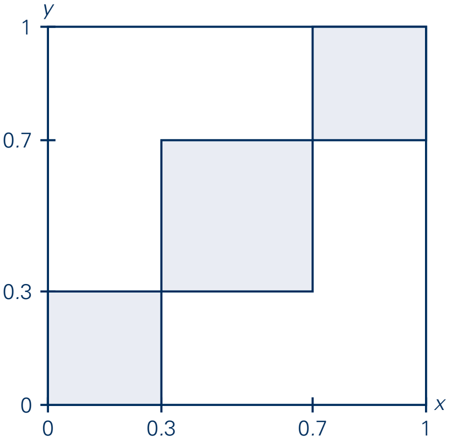

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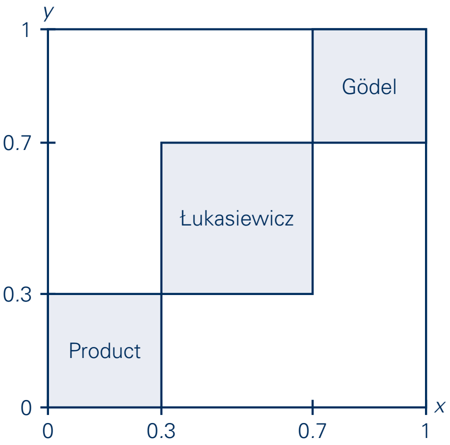

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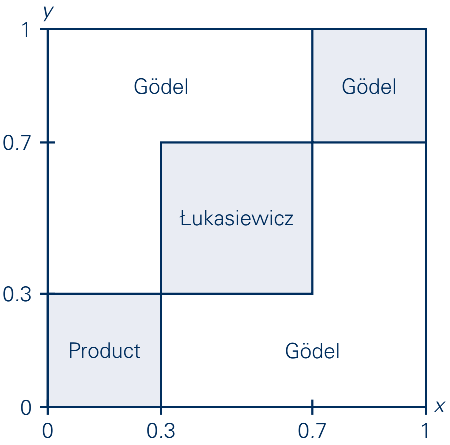

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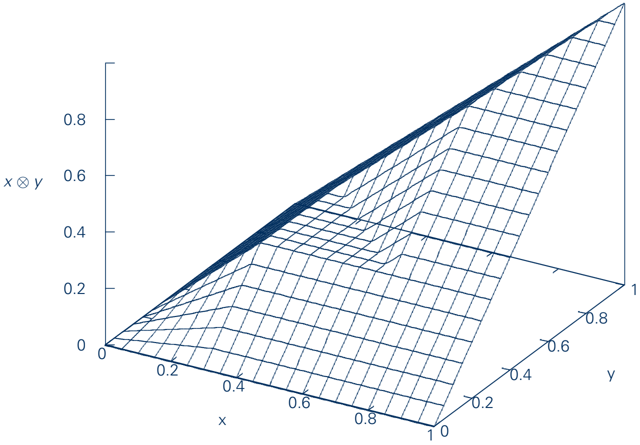


Isomorphic t-norms

If there is s is a bijective, monotone function

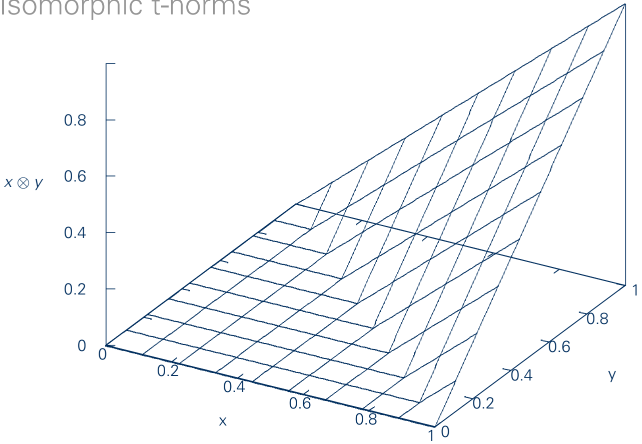
$$s\colon [0,1] \to [0,1]$$

satisfying

$$x \otimes_1 y = s^{-1}\big(s(x) \otimes_2 s(y)\big)$$

then $\otimes_1$ and $\otimes_2$ are called *isomorphic*.

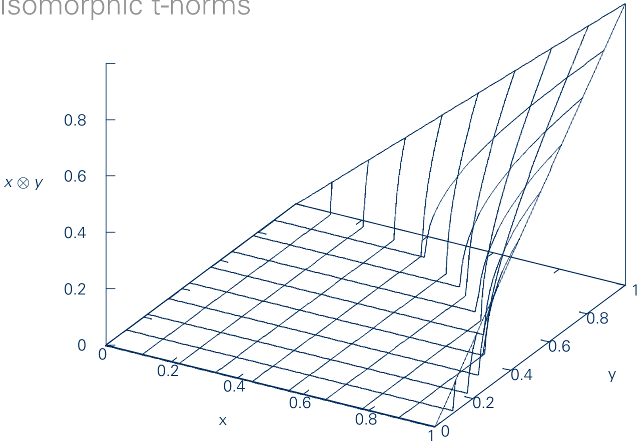
Isomorphic t-norms



Łukasiewicz t-norm (aka 1st Schweizer-Sklar t-norm)

$$x \otimes y = \max\{x + y - 1, 0\}$$

Isomorphic t-norms



2nd Schweizer-Sklar t-norm

$$x \otimes y = \sqrt{\max\{x^2 + y^2 - 1, 0\}}$$

Syntax

P countable set of propositional variables, $\otimes$ continuous t-norm.
Formulas of $PC(\otimes)$ are

- $\mathbf{0}$,
- p,
- $f_1 \mathbin{\&} f_2$, and
- $f_1 \to f_2$.

Semantics

Valuation $\mathcal{V}\colon P \to [0, 1]$

Zero	$\mathcal{V}(\mathbf{0}) = 0,$
Strong Conjunction	$\mathcal{V}(\phi \mathbin{\&} \psi) = \mathcal{V}(\phi) \otimes \mathcal{V}(\psi),$
Implication	$\mathcal{V}(\phi \to \psi) = \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\psi).$

Abbreviations

$$\begin{array}{lrl}
\text{Weak Conjunction} & \phi \wedge \psi & := \phi \,\&(\phi \to \psi), \\
\text{Weak Disjunction} & \phi \vee \psi & := \big((\phi \to \psi) \to \psi\big) \wedge \big((\psi \to \phi) \to \phi\big) \\
\text{Negation} & \neg\phi & := \phi \to \mathbf{0} \\
\text{Equivalence} & \phi \equiv \psi & := (\phi \to \psi) \,\&(\psi \to \phi) \\
\text{One} & \mathbf{1} & := \mathbf{0} \to \mathbf{0}.
\end{array}$$

1-Tautologies

Formula ϕ such that

$$\mathcal{V}(\phi) = \mathbf{1}$$

for every valuation $\mathcal{V}$.

Different t-Norms, Different 1-Tautologies

$$\neg\neg\phi \to \phi$$

Different t-Norms, Different 1-Tautologies

$$\neg\neg\phi \to \phi$$

- *Łukasiewicz:*

$$\begin{aligned}
\mathcal{V}(\neg\neg\phi \to \phi) &= \ominus \ominus \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\phi) \\
&= 1 - \big(1 - \mathcal{V}(\phi)\big) \Rightarrow \mathcal{V}(\phi) \\
&= \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\phi) \\
&= 1
\end{aligned}$$

1-tautology for Łukasiewicz

Different t-Norms, Different 1-Tautologies

$$\neg\neg\phi \to \phi$$

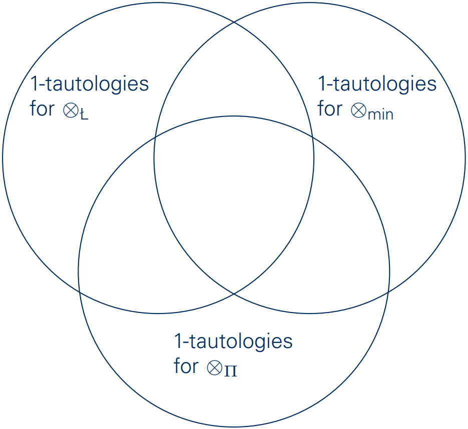
- *Łukasiewicz:*

$$\begin{aligned} \mathcal{V}(\neg\neg\phi \to \phi) &= \ominus \ominus \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\phi) \\ &= 1 - \big(1 - \mathcal{V}(\phi)\big) \Rightarrow \mathcal{V}(\phi) \\ &= \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\phi) \\ &= 1 \end{aligned}$$

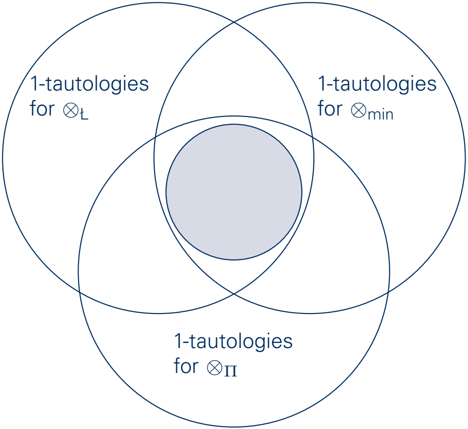
  1-tautology for Łukasiewicz
- *Gödel or Product:* Not a 1-tautology! Assume $\mathcal{V}(\phi) = 0.5$, then

$$\begin{aligned} \mathcal{V}(\neg\neg\phi \to \phi) &= \ominus \ominus \mathcal{V}(\phi) \Rightarrow \mathcal{V}(\phi) \\ &= \ominus 0 \Rightarrow \mathcal{V}(\phi) \\ &= 1 \Rightarrow 0.5 = 0.5 \end{aligned}$$

1-Tautologies for all t-Norms



1-Tautologies for all t-Norms



We are interested in 1-tautologies for all t-norms.

Axioms of Basic Logic

(A1) $(\phi \to \psi) \to \big((\psi \to \chi) \to (\phi \to \chi)\big)$

(A2) $\phi \,\&\, \psi \to \phi$

(A3) $\phi \,\&\, \psi \to \psi \,\&\, \phi$

(A4) $\phi \,\&(\phi \to \psi) \to \psi \,\&(\psi \to \phi)$

(A5) $\big(\phi \to (\psi \to \chi)\big) \to (\phi \,\&\, \psi \to \chi)$

(A6) $(\phi \,\&\, \psi \to \chi) \to \big(\phi \to (\psi \to \chi)\big)$

(A7) $((\phi \to \psi) \to \chi) \to \Big(((\psi \to \phi) \to \chi) \to \chi\Big)$

(A8) $\mathbf{0} \to \phi$

- *Deduction Rule:* Modus Ponens

Instantiation

Substitution arbitrary formulae for variable names, e.g.

$$(\rho \to \sigma) \,\&\, \psi \to (\rho \to \sigma)$$

is obtained from $\phi \,\&\, \psi \to \phi$ by substituting $\rho \to \sigma$ for ϕ.

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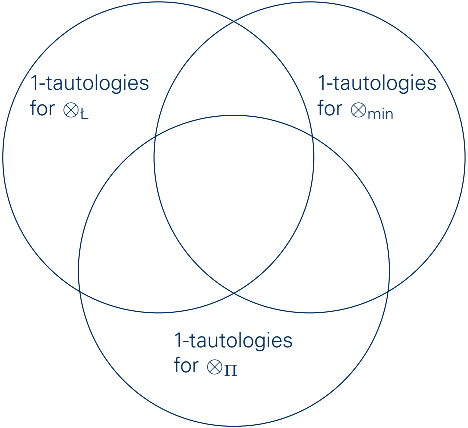
Application of Modus Ponens

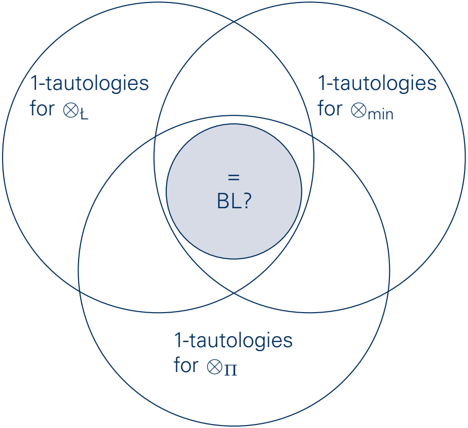
$BL \vdash \rho \,\&\, \sigma \to \sigma \,\&\, \rho$	Instance of (A3)
$BL \vdash (\rho \,\&\, \sigma \to \sigma \,\&\, \rho) \to \big((\sigma \,\&\, \rho \to \sigma) \to (\rho \,\&\, \sigma \to \sigma)\big)$	Instance of (A1)
$BL \vdash (\sigma \,\&\, \rho \to \sigma) \to (\rho \,\&\, \sigma \to \sigma)$	modus ponens
$BL \vdash \sigma \,\&\, \rho \to \sigma$	Instance of (A2)
$BL \vdash \rho \,\&\, \sigma \to \sigma$	modus ponens

Goal



1-tautologies for $\otimes_Ł$
1-tautologies for $\otimes_{\min}$
1-tautologies for $\otimes_\Pi$

Goal


1-tautologies
for $\otimes_Ł$
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=
BL?
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