Chapter 3

Basic Model Theory

Motivation

Interpretations in $\mathcal{ALC}$ can be seen as labeled graphs

We use bisimulations between graphs to study the expressive power of $\mathcal{ALC}$:

- $\mathcal{ALC}$ concepts cannot distinguish between bisimilar nodes
- models have the following properties:
  - tree model property ($\mathcal{T}$ has a model if it has a "tree-shaped" one)
  - closure under disjoint union

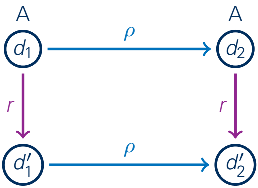
We also show that $\mathcal{ALC}$ has the finite model property

Definition 3.1

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two interpretations.

The relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ iff:

- if $d_1 \; \rho \; d_2$, then for every $A \in N_C$ it holds that $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$
- if $d_1 \; \rho \; d_2$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$, then there exists $d_2' \in \Delta^{\mathcal{I}_2}$ such that $d_1' \; \rho \; d_2'$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$
- if $d_1 \; \rho \; d_2$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$, then there exists $d_1' \in \Delta^{\mathcal{I}_1}$ such that $d_1' \; \rho \; d_2'$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$



Notice:

- it can be that $\mathcal{I}_1 = \mathcal{I}_2$
- the empty relation $\emptyset$ is always a bisimulation

Notation

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ interpretations, and $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$.

We write $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$ if there exists a bisimulation ρ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that

$$d_1 \; \rho \; d_2$$

Theorem 3.2 (bisimulation invariance of $\mathcal{ALC}$)

If $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$ then, for every concept C it holds that

$$d_1 \in C^{\mathcal{I}_1} \quad \text{iff} \quad d_2 \in C^{\mathcal{I}_2}$$

$\mathcal{ALC}$ concepts cannot distinguish between d_1 and d_2

Proof

blackboard

Expressive Power of $\mathcal{ALC}$

We have seen some extensions of $\mathcal{ALC}$ with new constructors:

- number restrictions
- nominals
- inverse roles

Question: are these really extensions of $\mathcal{ALC}$?

can they be expressed using just the constructors in $\mathcal{ALC}$?

We construct concepts that cannot be expressed in $\mathcal{ALC}$ (but can in the extensions)

$\mathcal{ALC}$ vs. $\mathcal{ALCN}$

Proposition 3.3 ($\mathcal{ALCN}$ is more expressive than $\mathcal{ALC}$)

No $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCN}$ concept

$$(\leq 1\ r)$$

Proof

blackboard

Proposition 3.4 ($\mathcal{ALCI}$ is more expressive than $\mathcal{ALC}$)

No $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCI}$ concept

$$\exists r^{-1}.\top$$

Proof

blackboard

Proposition 3.5 ($\mathcal{ALCO}$ is more expressive than $\mathcal{ALC}$)

No $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCO}$ concept

$$\{a\}$$

Proof

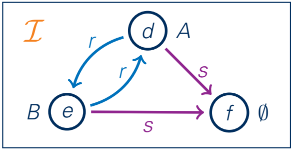
blackboard

Tree Model Property

Recall: interpretations can be seen as graphs:

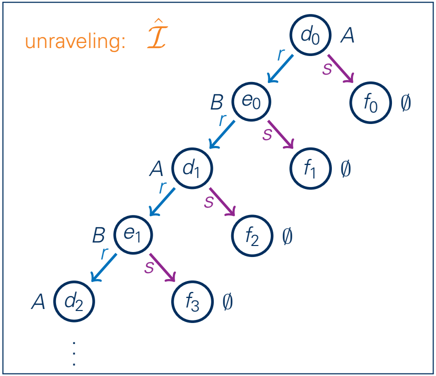
- $\Delta^{\mathcal{I}}$ defines the nodes
- interpretation of roles defines the edges
- interpretation of concept names yield labels

Starting from a given node, the graph can be unraveled into a tree without violating the axioms



$\mathcal{I}$ is a model of the TBox:

$$A \sqsubseteq \exists r.B$$
$$B \sqsubseteq \exists r.A$$
$$A \sqcup B \sqsubseteq \exists r.\top$$



Definition 3.6 (tree model)

Let $\mathcal{T}$ be a TBox and C a concept.

The interpretation $\mathcal{I}$ is a tree model of C w.r.t. $\mathcal{T}$ iff $\mathcal{I}$ is a model of $\mathcal{T}$ and the graph

$$(\Delta^{\mathcal{I}}, \bigcup_{r \in N_R} r^{\mathcal{I}})$$

is a tree whose root belongs to $C^{\mathcal{I}}$

Theorem 3.7 (tree model property)

$\mathcal{ALC}$ has the tree model property:

if the concept C is satisfiable w.r.t. the TBox $\mathcal{T}$, then C has a tree model w.r.t. $\mathcal{T}$

Proof

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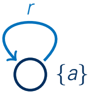
Proposition 3.8 ($\mathcal{ALCO}$ lacks the tree model property)

$\mathcal{ALCO}$ does not have the tree model property

Proof

The concept $\{a\}$ does not have a tree model w.r.t. $\{\{a\} \sqsubseteq \exists r.\{a\}\}$

$\mathcal{I}$:



Definition 3.9

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two interpretations over disjoint domains $(\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{I}_2} = \emptyset)$

Their disjoint union $\mathcal{I}_1 \uplus \mathcal{I}_2$ is the interpretation defined by:

- $\Delta^{\mathcal{I}_1 \uplus \mathcal{I}_2} = \Delta^{\mathcal{I}_1} \cup \Delta^{\mathcal{I}_2}$
- $A^{\mathcal{I}_1 \uplus \mathcal{I}_2} = A^{\mathcal{I}_1} \cup A^{\mathcal{I}_2}$ for all $A \in N_C$
- $r^{\mathcal{I}_1 \uplus \mathcal{I}_2} = r^{\mathcal{I}_1} \cup r^{\mathcal{I}_2}$ for all $r \in N_R$

Notice: for all concepts C and $d \in \Delta^{\mathcal{I}_i}$ $(i \in \{1, 2\})$, we have

$$d \in C^{\mathcal{I}_1} \cup C^{\mathcal{I}_2} \quad \text{iff} \quad d \in C^{\mathcal{I}_1 \uplus \mathcal{I}_2}$$

Theorem 3.10

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two interpretations over disjoint domains.
If they are both models of $\mathcal{T}$, then $\mathcal{I}_1 \uplus \mathcal{I}_2$ is also a model of $\mathcal{T}$

Proof

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Definition 3.11 (finite model)

Let $\mathcal{T}$ be a TBox and C a concept.

The interpretation $\mathcal{I}$ is a finite model of C w.r.t. $\mathcal{T}$ iff

- $\mathcal{I}$ is a model of $\mathcal{T}$,
- $C^{\mathcal{I}} \neq \emptyset$, and
- $\Delta^{\mathcal{I}}$ is finite

Theorem 3.12 (finite model property)

$\mathcal{ALC}$ has the finite model property:

if the concept C is satisfiable w.r.t. the TBox $\mathcal{T}$, then C has a finite model w.r.t. $\mathcal{T}$

Before proving this, we need some definitions

Size of Concepts

The size of concepts is defined inductively:

- $|A| := 1$ for all $A \in N_C$
- if $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$, then $|C| := |C_1| + |C_2| + 1$
- if $C = \neg D$, $C = \exists r.D$, or $C = \forall r.D$, then $|C| := |D| + 1$

$$\begin{aligned} |A \sqcap \exists r.(A \sqcup B)| &= |A| + |\exists r.(A \sqcup B)| + 1 \\ &= 1 + |A \sqcup B| + 1 + 1 \\ &= 3 + |A| + |B| + 1 = 6 \end{aligned}$$

count the number of occurrences of concept names, role names, and Boolean operators

$$|\mathcal{T}| := \sum_{C \sqsubseteq D \in \mathcal{T}} |C| + |D|$$

Subconcepts

The set of subconcepts of a concept is defined inductively:

- $\mathsf{sub}(A) := \{A\}$ for all $A \in N_C$
- if $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$, then $\mathsf{sub}C := \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2) \cup \{C\}$
- if $C = \neg D$, $C = \exists r.D$, or $C = \forall r.D$, then $\mathsf{sub}(C) := \mathsf{sub}(D) \cup \{C\}$

$$\mathsf{sub}(A \sqcap \exists r.(A \sqcup B)) = \{A \sqcap \exists r.(A \sqcup B),\ A,\ \exists r.(A \sqcup B),\ A \sqcup B,\ B\}$$

$$\mathsf{sub}(\mathcal{T}) := \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(C) \cup \mathsf{sub}(D)$$

Note:

- the cardinality of $\mathsf{sub}(C)$ is bounded by $|C|$
- the cardinality of $\mathsf{sub}(\mathcal{T})$ is bounded by $|\mathcal{T}|$

Definition 3.13 (S-type)

Let S be a finite set of concepts, $\mathcal{I}$ an interpretation, and $d \in \Delta^{\mathcal{I}}$.

The S-type of d is defined as:

$$t_S(d) := \{C \in S \mid d \in C^{\mathcal{I}}\}.$$

Lemma 3.14 (number of types)

$$|\{t_S(d) \mid d \in \Delta^{\mathcal{I}}\}| \leq 2^{|S|}$$

Proof

obvious

Filtration of Models

Let S be a finite set of concepts, $\mathcal{I}$ an interpretation, and $d \in \Delta^{\mathcal{I}}$.

We define the equivalence relation $\simeq$ on $\Delta^{\mathcal{I}}$ as follows

$$d \simeq e \quad \text{iff} \quad t_S(d) = t_S(e)$$

The $\simeq$-equivalence class of d is denoted by $[d]$

Definition 3.15 (S-filtration)

The S-filtration of $\mathcal{I}$ is the interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ where

- $\Delta^{\mathcal{J}} := \{[d] \mid d \in \Delta^{\mathcal{I}}\}$,
- $A^{\mathcal{J}} := \{[d] \mid \exists d' \in [d].d' \in A^{\mathcal{I}}\}$ for all $A \in N_C$, and
- $r^{\mathcal{J}} := \{([d], [e]) \mid \exists d' \in [d], e' \in [e].(d', e') \in r^{\mathcal{I}}\}$ for all $r \in N_R$

Obviously, $|\Delta^{\mathcal{I}}| \leq 2^{|S|}$

Filtration on Closed Sets

We say that a finite set S of concepts is closed iff

$$\bigcup_{C \in S} \mathsf{sub}(C) \subseteq S$$

Lemma 3.16

Let S be a closed finite set of concepts, $\mathcal{I}$ an interpretation, and $\mathcal{J}$ the S-filtration of $\mathcal{I}$.

For all $d \in \Delta^{\mathcal{I}}$ and all $C \in S$ it holds that

$$d \in C^{\mathcal{I}} \quad \text{iff} \quad [d] \in C^{\mathcal{J}}$$

Proof

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Bounded Model Property

We show a property that is stronger than the finite model property.

Theorem 3.17

Let $\mathcal{T}$ be an $\mathcal{ALC}$-TBox, C an $\mathcal{ALC}$-concept, and $S := \mathsf{sub}(C) \cup \mathsf{sub}(\mathcal{T})$

If C is satisfiable w.r.t. $\mathcal{T}$ then there is a model $\hat{\mathcal{I}}$ of $\mathcal{T}$ such that:

- $C^{\hat{\mathcal{I}}} \neq \emptyset$, and
- $|\Delta^{\hat{\mathcal{I}}}| \leq 2^{|S|}$

Proof

Let $\mathcal{I}$ be a model of $\mathcal{T}$ with $C^{\mathcal{I}} \neq \emptyset$ and $\hat{\mathcal{I}}$ the S-filtration of $\mathcal{I}$.

We must show:

- $\Delta^{\hat{\mathcal{I}}} \leq 2^{|S|}$ (Lemma 3.14)
- $C^{\hat{\mathcal{I}}} \neq \emptyset$
- $\hat{\mathcal{I}}$ is a model of $\mathcal{T}$ (Lemma 3.16)

Bounded Model Property

We show a property that is stronger than the finite model property.

Theorem 3.17

Let $\mathcal{T}$ be an $\mathcal{ALC}$-TBox, C an $\mathcal{ALC}$-concept, and $S := \text{sub}(C) \cup \text{sub}(\mathcal{T})$

If C is satisfiable w.r.t. $\mathcal{T}$ then there is a model $\hat{\mathcal{I}}$ of $\mathcal{T}$ such that:

- $C^{\hat{\mathcal{I}}} \neq \emptyset$, and
- $|\Delta^{\hat{\mathcal{I}}}| \leq 2^{|S|}$

Corollary 3.18 (decidability)

In $\mathcal{ALC}$, satisfiability of concepts w.r.t. a TBox is decidable

Theorem 3.19 ($\mathcal{ALCNI}$ lacks the finite model property)
$\mathcal{ALCNI}$ does not have the finite model property

Proof
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