Exercise 40

Determine whether Player 2 has a winning strategy in the PSPACE game $G = (\phi, \{p_0, p_2\}, \{p_1, p_3\}, <)$ with

$$\phi = (\neg p_0 \rightarrow p_1) \land ((p_0 \land p_1) \rightarrow (p_2 \lor p_3)) \land (\neg p_1 \rightarrow (p_3 \rightarrow \neg p_2))$$

and $p_i < p_j$ iff $i < j$.

Exercise 41

A *quantified Boolean formula* is of the form $\phi = Q_1 p_1 \ldots Q_n p_n \psi$ where $Q_1, \ldots, Q_n \in \{\forall, \exists\}$, $p_1, \ldots, p_n$ are propositional variables, and $\psi$ is propositional formula containing only the variables $p_1, \ldots, p_n$.

Such a formula is *valid* iff the following is satisfied:

- For $n = 0$, the formula $\phi$ does not contain variables, and thus is a Boolean combination of 0 and 1. It is valid iff this Boolean combination of 0 and 1 evaluates to 1.

- For $n > 0$, we consider:

  $$\phi_0 := Q_2 p_2 \ldots Q_n p_n \psi[p_1 := 0], \text{ and}$$
  $$\phi_1 := Q_2 p_2 \ldots Q_n p_n \psi[p_1 := 1].$$

  If $Q_1 = \exists$, then $\phi$ is valid iff one of $\phi_0$ and $\phi_1$ is valid.
  If $Q_1 = \forall$, then $\phi$ is valid iff both $\phi_0$ and $\phi_1$ are valid.

QBF denotes the set of valid quantified Boolean formulae. Prove by reduction of QBF that the problem of deciding the existence of winning strategy for Player 2 in PSPACE games is PSPACE-hard.

Exercise 42

Finish the proof of Lemma 6.7 by showing that $v_0 \in C^T_G$.
Exercise 43
Determine whether Player 2 has a winning strategy in the \( \text{EXPTime} \) game \( G = (\phi, \Gamma_1, \Gamma_2, t_0) \) with

- \( \phi = (p_1 \land p_2 \land p_3 \land \neg q) \lor (\neg p_1 \land \neg p_2 \land \neg p_3 \land q) \),
- \( \Gamma_1 = \{p_1, p_2, p_3\} \),
- \( \Gamma_2 = \{q\} \),
- \( t_0(p_1) = t_0(p_2) = t_0(p_3) = t_0(q) = 0 \).