Exercise 4.1 Prove or refute the following claim: If an \textit{ALC}-concept \( C \) is satisfiable w.r.t. an \textit{ALC}-TBox \( T \), then for all \( n \geq 1 \) there is a finite model \( I_n \) of \( T \) such that \( |C_{I_n}| \geq n \).

Does the claim hold if the condition “\( |C_{I_n}| \geq n \)” is replaced by “\( |C_{I_n}| = n \)”?

Exercise 4.2 Prove or refute the following claim: Given an \textit{ALC}-concept \( C \) and an \textit{ALC}-TBox \( T \), if \( I \) is an interpretation and \( J \) its filtration w.r.t. \( \text{sub}(C) \cup \text{sub}(T) \) (see Definition 3.14), then the relation \( \rho = \{(d, [d]) \mid d \in \Delta^I\} \) is a bisimulation between \( I \) and \( J \).

Exercise 4.3 We consider bisimulations between an interpretation \( I \) and itself, which are called bisimulations on \( I \). For two elements \( d, e \in \Delta^I \), we write \( d \approx_I e \) if they are bisimilar, i.e., if there is a bisimulation \( \rho \) on \( I \) such that \( d \rho e \).

(a) Show that \( \approx_I \) is an equivalence relation on \( \Delta^I \).
(b) Show that \( \approx_I \) is a bisimulation on \( I \).
(c) Show that, for finite interpretations \( I \), the relation \( \approx_I \) can be computed in time polynomial in the cardinality of \( I \).

Consider the interpretation \( J \) that is defined like the filtration (Definition 3.14), but with \( \approx_I \) instead of \( \approx \).

(d) Show that \( \rho = \{(d, [d]_{\approx_I}) \mid d \in \Delta^I\} \) is a bisimulation between \( I \) and \( J \).
(e) Show that, if \( I \) is a model of an \textit{ALC}-concept \( C \) w.r.t. an \textit{ALC}-TBox \( T \), then so is \( J \).
(f) Why can we not use the previous result to show the finite model property for \textit{ALC}?

Exercise 4.4 For the following interpretation \( I \), draw the unraveling of \( I \) at \( d \) up to depth 5, i.e., restricted to \( d \)-paths of length at most 5 (see Definition 3.21):

![Diagram](image)

Exercise 4.5 Prove or refute the following claim: If \( \mathcal{K} \) is an \textit{ALC}-knowledge base and \( C \) an \textit{ALC}-concept description such that \( C \) is satisfiable w.r.t. \( \mathcal{K} \), then \( C \) has a tree model w.r.t. \( \mathcal{K} \).