Description Logic

Franz Baader

Literature:

- Franz Baader, Carsten Lutz, Ian Horrocks, and Uli Sattler: An Introduction to Description Logic. Cambridge University Press, 2017.

- F. Baader, D. Calvanese, D. McGuinness, D. Nardi, P. Patel-Schneider (ed.): The Description Logic Handbook. Cambridge University Press, 2003.

- F. Baader, C. Lutz: Description Logics. In The Handbook of Modal Logic, Elsevier, 2006.

- F. Baader: Description Logics. In Reasoning Web: Semantic Technologies for Information Systems, 5th International Summer School 2009, Springer LNCS 5689, 2009.

Organisation

The lecture is accompanied by weekly tutorials held by Dr. Stefan Borgwardt.
The lectures and tutorials take place in room APB/E005 on

- Tuesdays 16:40–18:10,
- Wednesdays 14:50–16:20,
- Thursdays 16:40–18:10.

The distribution of lectures and tutorials can be found in the table on
`https://lat.inf.tu-dresden.de/teaching/ws2017-2018/DL/`.

During the semester, there may be changes to this schedule on short notice.
Be sure to check this page regularly for updates.

Exercise sheets will be made available on the Web page approximately one week before each tutorial.

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Chapter 1: Introduction

Description Logic

subfield of knowledge representation,

which is a subfield of Artificial Intelligence

Description comes from concept description, i.e., a formal expression that determines a set of individuals with common properties

Logics comes from the fact that the semantics of concept descriptions can be defined using logic;

in particular, most Description Logics can be seen as fragments of first-order logic.

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Description Logic

subfield of knowledge representation,
which is a subfield of Artificial Intelligence

Description Logic: name of a research field

Description Logics: a family of knowledge representation languages

Description Logic: a member of this family

DL(s)

Knowledge Representation

"develop formalisms for providing high-level descriptions of the world that can be effectively used to build intelligent applications"

[Brachman & Nardi, 2003]

- formalism: well-defined syntax and formal, unambiguous semantics
- high-level description: only relevant aspects represented, others left out
- intelligent applications: must be able to reason about the knowledge, and infer implicit knowledge from the explicitly represented knowledge
- effectively used: need for practical reasoning tools and efficient implementations

Syntax

- provides an explicit symbolic representation of the knowledge
- not just implicit, as e.g. in neural networks

$$
\begin{array}{lll}
\mathtt{Woman} & \equiv \mathtt{Person} \sqcap \mathtt{Female} & \\
\mathtt{Man} & \equiv \mathtt{Person} \sqcap \neg\mathtt{Female} & \\
\mathtt{Mother} & \equiv \mathtt{Woman} \sqcap \exists \mathtt{hasChild}.\top & \\
\mathtt{Person} & \equiv \mathtt{Man} \sqcup \mathtt{Woman} & \mathtt{Male(JOHN)} \\
\bot & \equiv \mathtt{Male} \sqcap \mathtt{Female} & \mathtt{Male(MARC)} \\
 & & \mathtt{Male(STEPHEN)} \\
\mathtt{hasChild(STEPHEN,MARC)} & & \mathtt{Male(JASON)} \\
\mathtt{hasChild(MARC,ANNA)} & & \mathtt{Female(MICHELLE)} \\
\mathtt{hasChild(JOHN,MARIA)} & & \mathtt{Female(ANNA)} \\
\mathtt{hasChild(ANNA,JASON)} & & \mathtt{Female(MARIA)}
\end{array}
$$

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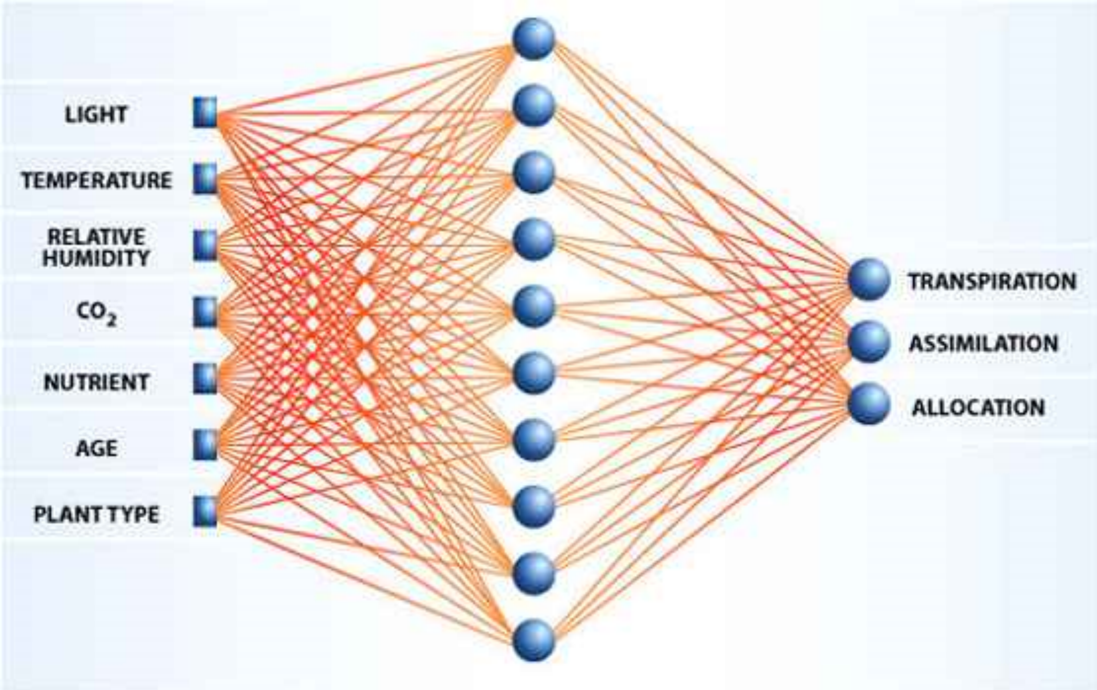
Syntax

- provides an explicit symbolic representation of the knowledge
- not just implicit, as e.g. in neural networks



Semantics

describes the connection between the symbolic representation and the real-world entities it is supposed to represent

- no procedural semantics, i.e., should not just be defined by how certain programs using the symbolic representation behave

- instead declarative semantics:
  - mapping of the symbolic expressions to an abstraction of the "world" (interpretation)
  - notion of "truth" that allows us to determine whether a symbolic expression is true in the world under consideration (model)

Syntax & Semantics

determine the expressive power of a formalism

Adequate expressive power:

- not too low: can all the knowledge relevant for solving the problem at hand be represented?
- not too high: are the available representational means really necessary in this application?

Reasoning

deduce implicit knowledge from the explicitly represented knowledge

$$\forall x.\forall y.\,(male(y) \wedge \exists z.(has\_child(x,z) \wedge has\_child(z,y)) \rightarrow has\_grandson(x,y)$$

$has\_child(John, Mary)$

$has\_child(Mary, Paul)$

$male(Paul)$

$grandson\_of(John, Paul)$

implicit knowledge

Knowledge representation systems

should provide their users with inference tools

that can deduce (certain) implicit consequences

results should depend only on the semantics of the representation language,

and not on the syntactic representation:

semantically equivalent knowledge should lead to the same result

Reasoning

deduce implicit knowledge from the explicitly represented knowledge

$$\forall x.\forall y.\forall z.\,(has\_child(x,z) \wedge has\_child(z,y) \wedge male(y)) \rightarrow has\_grandson\_of(x,y)$$

$has\_child(John, Mary)$

$has\_child(Mary, Paul)$

$male(Paul)$

$grandson\_of(John, Paul)$

implicit knowledge

Knowledge representation systems

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results should depend only on the semantics of the representation language,

and not on the syntactic representation:

semantically equivalent knowledge should lead to the same result

Reasoning procedures

- The procedure should be a decision procedure for the problem:
  - soundness: positive answers are correct
  - completeness: negative answers are correct
  - termination: always gives an answer in finite time
- The procedure should be as efficient as possible:
  preferably optimal w.r.t. the (worst-case) complexity of the problem
- The procedure should be practical:
  easy to implement and optimize, and behave well in applications

Reasoning procedures

- Satisfiability in first-order logic does not have a decision procedure.
  - full first-order logic is thus not an appropriate knowledge representation formalism
- Satisfiability in propositional logic has a decision procedure, but the problem is NP-complete.
  - there are, however, highly optimized SAT solvers that behave well in practice
  - expressive power is, however, often not sufficient to express the relevant knowledge

Terminological knowledge

formalize the terminology of the application domain:

- define important notions (classes, relations, objects) of the domain
- state constraints on the way these notions can be interpreted
- deduce consequences of definitions and constraints:
  subclass relationships, instance relationships

Example: domain conference

- classes (concepts) like Person, Speaker, Author, Talk, Participant, Workshop, . . .
- relations (roles) like gives, attends, likes, . . .
- objects (individuals) like Richard, Frank, Paper_176, . . .
- constraints like: every talk is given by a speaker, every speaker is an author, every workshop must have at least 10 participants, . . .

Ontologies

terminological knowledge bases are nowadays often called ontologies

Ontologies are, for example, used in:

- the Semantic Web to enable a common understanding of important notions, which can be used in the semantic labeling of Web pages
- Information Retrieval to support the automatic extraction of information from text documents
- Medicine to provide formal definitions for important notions that can be used by medical doctors to describe findings and procedures, insurance companies to determine payment, . . . (SNOMED CT, GALEN, . . .)
- Biology to enable semantic access to gene databases (Gene Ontology)
- . . .

Description Logics

class of logic-based knowledge representation formalisms tailored towards representing terminological knowledge

Prehistory:

- Descended from early approaches for representing terminological knowledge
  - semantic networks (Quillian, 1968)
  - frames (Minsky, 1975)

- problems with missing semantics lead to
  - structured inheritance networks (Brachman, 1978)
  - the first DL system KL-ONE (Brachman&Schmolze, 1985)

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Phase 1:

- implementation of systems (Back, K-Rep, Loom, Meson, . . .)
- based on incomplete structural subsumption algorithms

Phase 2:

- development of tableau-based algorithms and complexity results
- first implementation of tableau-based systems (Kris, Crack)
- first formal investigation of optimization methods

Phase 3:

- tableau-based algorithms for very expressive DLs
- highly optimized tableau-based systems (FaCT, Racer, HermiT, Konclude, ...)
- relationship to modal logic and decidable fragments of FOL

Phase 4:

- Web Ontology Language (OWL-DL) based on very expressive DL
- industrial-strength reasoners and ontology editors for OWL-DL
- investigation of light-weight DLs with tractable reasoning problems
- query answering w.r.t. ontologies for large data sets

Chapter 2

A Basic Description Logic

$\mathcal{ALC}$ attributive language with complement
[Schmidt-Schauß&Smolka, 1991]

Naming scheme:

- basic language $\mathcal{AL}$
- extended with contructors whose "letter" is added after the $\mathcal{AL}$
- $\mathcal{C}$ stands for complement, i.e., $\mathcal{ALC}$ is obatined from $\mathcal{AL}$ by adding the complement ($\neg$) operator

Description logic system

structure

2.1

description language

- constructors for building complex concepts out of atomic concepts and roles
- formal, logic-based semantics

TBox 2.2

defines the terminology of the application domain

ABox 2.3

states facts about a specific "world"

knowledge base

2.4

reasoning component

- derive implicitly respresented knowledge (e.g., subsumption)
- "practical" algorithms

2.1. The description language

syntax and semantics of $\mathcal{ALC}$

Definition 2.1 (Syntax of $\mathcal{ALC}$)

Let **C** and **R** be disjoint sets of concept names and role names, respectively.

$\mathcal{ALC}$-concept descriptions are defined by induction:

- If $A \in \mathbf{C}$, then A is an $\mathcal{ALC}$-concept description.
- If C, D are $\mathcal{ALC}$-concept descriptions, and $r \in \mathbf{R}$, then the following are $\mathcal{ALC}$-concept descriptions:
  - $C \sqcap D$ (conjunction)
  - $C \sqcup D$ (disjunction)
  - $\neg C$ (negation)
  - $\forall r.C$ (value restriction)
  - $\exists r.C$ (existential restriction)

Abbreviations:

- $\top := A \sqcup \neg A$ (top)
- $\bot := A \sqcap \neg A$ (bottom)
- $C \Rightarrow D := \neg C \sqcup D$ (implication)

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Notation (use and abuse):

- concept names are called atomic
- all other descriptions are called complex
- instead of $\mathcal{ALC}$-concept description we often say $\mathcal{ALC}$-concept or concept description or concept
- A, B often used for concept names, C, D for complex concept descriptions, r, s for role names

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The description language

examples of $\mathcal{ALC}$-concept descriptions

$\text{Person} \sqcap \text{Female}$

$\text{Participant} \sqcap \exists \text{attends}.\text{Talk}$

$\text{Participant} \sqcap \forall \text{attends}.(\text{Talk} \sqcap \neg \text{Boring})$

$\text{Speaker} \sqcap \exists \text{gives}.(\text{Talk} \sqcap \forall \text{topic}.\text{DL})$

$\text{Speaker} \sqcap \forall \text{gives}.(\text{Talk} \sqcap \exists \text{topic}.(\text{DL} \sqcup \text{FuzzyLogic}))$

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Definition 2.2 (Semantics of $\mathcal{ALC}$)

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty domain $\Delta^{\mathcal{I}}$ and an extension mapping $\cdot^{\mathcal{I}}$:

- $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for all $A \in \mathbf{C}$, — concepts interpreted as sets
- $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for all $r \in \mathbf{R}$. — roles interpreted as binary relations

The extension mapping is extended to complex $\mathcal{ALC}$-concept descriptions as follows:

- $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$
- $(C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}$
- $(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
- $(\forall r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \text{for all } e \in \Delta^{\mathcal{I}} : (d,e) \in r^{\mathcal{I}} \text{ implies } e \in C^{\mathcal{I}}\}$
- $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \text{there is } e \in \Delta^{\mathcal{I}} : (d,e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\}$

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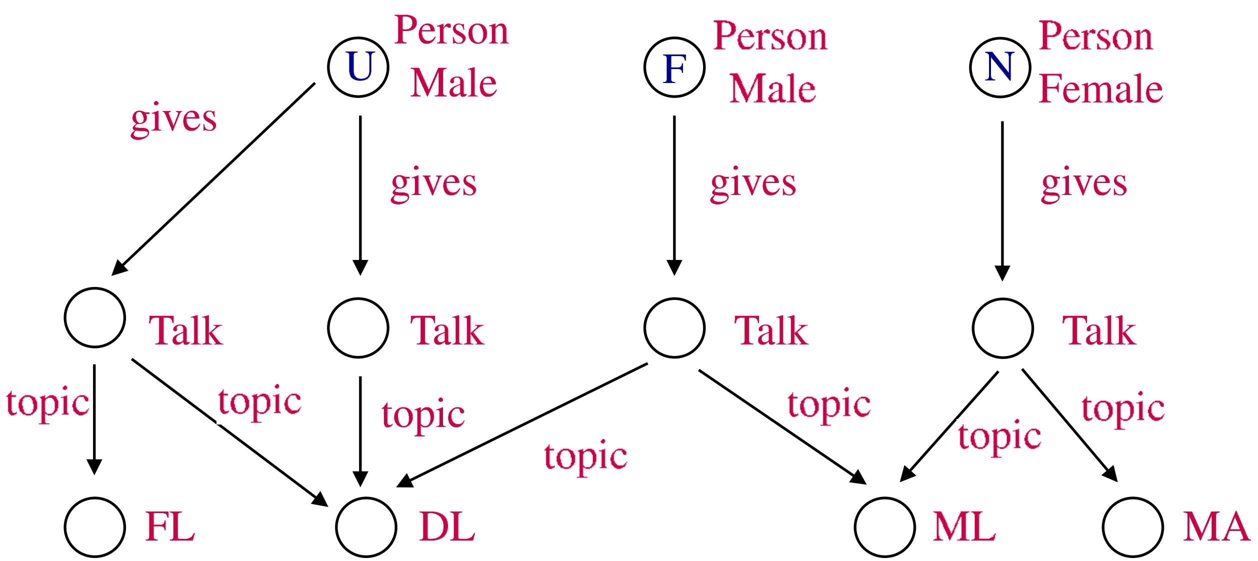
Example

of an interpretation



$$\text{Person} \sqcap \exists\text{gives}.(\text{Talk} \sqcap \forall\text{topic}.\text{DL})$$

$$\text{Person} \sqcap \forall\text{gives}.(\text{Talk} \sqcap \exists\text{topic}.\text{DL})$$

Relationship with First-Order Logic

$\mathcal{ALC}$ can be seen as a fragment of first-order logic:

- Concept names are unary predicates, and role names are binary predicates.
- Interpretations for $\mathcal{ALC}$ can then obviously be viewed as first-order interpretations for this signature.
- Concept descriptions correspond to first-order formulae with one free variable.
- Given such a formula $\phi(x)$ with the free variable x and an interpretation $\mathcal{I}$, the extension of ϕ w.r.t. $\mathcal{I}$ is given by

$$\phi^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \mathcal{I} \models \phi(d)\}$$

- Goal: translate $\mathcal{ALC}$-concepts C into first-order formulae $\tau_x(C)$ such that their extensions coincide.

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Relationship with First-Order Logic

Concept description C translated into formula with one free variable $\tau_x(C)$:

- $\pi_x(A) := A(x)$ for $A \in \mathbf{C}$
- $\pi_x(C \sqcap D) := \pi_x(C) \wedge \pi_x(D)$
- $\pi_x(C \sqcup D) := \pi_x(C) \vee \pi_x(D)$
- $\pi_x(\neg C) := \neg \pi_x(C)$
- $\pi_x(\forall r.C) := \forall y.(r(x,y) \rightarrow \pi_y(C))$
- $\pi_x(\exists r.C) := \exists y.(r(x,y) \wedge \pi_y(C))$

y variable different from x

$$\begin{aligned}\pi_x(\forall r.(A \sqcap \exists r.B)) &= \forall y.(r(x,y) \rightarrow \pi_y(A \sqcap \exists r.B)) \\ &= \forall y.(r(x,y) \rightarrow (A(y) \wedge \exists z.(r(y,z) \wedge B(z))))\end{aligned}$$

Relationship with First-Order Logic

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y variable different from x

Lemma 2.3

C and $\pi_x(C)$ have the same extension, i.e.,

$$C^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \mathcal{I} \models \pi_x(C)(d)\}$$

Proof: induction on the structure of C

Relationship with First-Order Logic

$\mathcal{ALC}$ can be seen as a fragment of first-order logic:

- Concept names are unary predicates, and role names are binary predicates.
- Concept descriptions C yield formulae with one free variable $\tau_x(C)$.

These formulae belong to known decidable subclasses of first-order logic:

– two-variable fragment

– guarded fragment

$$\begin{aligned}\pi_x(\forall r.(A \sqcap \exists r.B)) &= \forall y.(r(x,y) \rightarrow \pi_y(A \sqcap \exists r.B)) \\ &= \forall y.(r(x,y) \rightarrow (A(y) \wedge \exists z.(r(y,z) \wedge B(z))))\end{aligned}$$

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Relationship with Modal Logic

multimodal K:

several pairs of boxes and diamonds

$\mathcal{ALC}$ is a syntactic variant of the basic modal logic K:

- Concept names are propositional variables, and role names are names for transition relations.
- Concept descriptions C yield modal formulae $\pi(C)$:
  - $\pi(A) := a$ for $A \in \mathbf{C}$
  - $\pi(C \sqcap D) := \pi(C) \wedge \pi(D)$
  - $\pi(C \sqcup D) := \pi(C) \vee \pi(D)$
  - $\pi(\neg C) := \neg\pi(C)$
  - $\pi(\forall r.C) := [r]\pi(C)$
  - $\pi(\exists r.C) := \langle r\rangle\pi(C)$

C and $\pi(C)$ have the same semantics: $C^{\mathcal{I}}$ is the set of worlds that make $\pi(C)$ true in the Kripke structure described by $\mathcal{I}$.

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Additional constructors

$\mathcal{ALC}$ is only an example of a description logic.

DL researchers have introduced and investigated many additional constructors.

Example letter $\mathcal{Q}$ in the naming scheme

Qualified number restrictions: $(\geq n\, r.C)$, $(\leq n\, r.C)$ with semantics

$$(\geq n\, r.C)^{\mathcal{I}} \quad := \quad \{d \in \Delta^{\mathcal{I}} \mid \mathsf{card}(\{e \mid (d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\}) \geq n\}$$

$$(\leq n\, r.C)^{\mathcal{I}} \quad := \quad \{d \in \Delta^{\mathcal{I}} \mid \mathsf{card}(\{e \mid (d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\}) \leq n\}$$

Persons that attend at most 20 talks, of which at least 3 have the topic DL:

$$\mathsf{Person} \sqcap (\leq 20\, \mathsf{attends.Talk}) \sqcap (\geq 3\, \mathsf{attends.}(\mathsf{Talk} \sqcap \exists \mathsf{topic.DL}))$$

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(\leq n\, r.C)^{\mathcal{I}} & := & \{d \in \Delta^{\mathcal{I}} \mid \text{card}(\{e \mid (d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\}) \leq n\}
\end{array}
$$

Number restrictions: $(\geq n\, r)$, $(\leq n\, r)$ as abbreviation for $(\geq n\, r.\top)$ and $(\leq n\, r.\top)$:

$$
\begin{array}{lcl}
(\geq n\, r)^{\mathcal{I}} & := & \{d \in \Delta^{\mathcal{I}} \mid \text{card}(\{e \mid (d,e) \in r^{\mathcal{I}}\}) \geq n\} \\
(\leq n\, r)^{\mathcal{I}} & := & \{d \in \Delta^{\mathcal{I}} \mid \text{card}(\{e \mid (d,e) \in r^{\mathcal{I}}\}) \leq n\}
\end{array}
$$

letter $\mathcal{N}$ in the naming scheme

Additional constructors

In addition to concept constructors, one can also introduce role constructors.

Example

letter $\mathcal{I}$ in the naming scheme

Inverse roles: if r is a role, then r^- denotes its inverse

$$(r^-)^\mathcal{I} := \{(e,d) \mid (d,e) \in r^\mathcal{I}\}$$

Inverse roles can be used like role names in value and existential restrictions.

Presenter of a boring talk:

$$\text{Speaker} \sqcap \exists \text{gives}.(\text{Talk} \sqcap \forall \text{attends}^-.(\text{Bored} \sqcup \text{Sleeping}))$$

Description logic system

structure

2.1

description language

- constructors for building complex concepts out of atomic concepts and roles
- formal, logic-based semantics

TBox 2.2

defines the terminology of the application domain

ABox 2.3

states facts about a specific “world”

knowledge base

2.4

reasoning component

- derive implicitly respresented knowledge (e.g., subsumption)
- “practical” algorithms

Definition 2.4 (GCIs and TBoxes)

- A general concept inclusion is of the form $C \sqsubseteq D$ where C, D are concept descriptions.
- A TBox is a finite set of GCIs.
- The interpretation $\mathcal{I}$ satisfies the GCI $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- The interpretation $\mathcal{I}$ is a model of the TBox $\mathcal{T}$ iff it satisfies all the GCIs in $\mathcal{T}$.

Note: this definition is not specific for $\mathcal{ALC}$.
It applies also to other concept description languages.

2.2. Terminological knowledge

Definition 2.4 (GCIs and TBoxes)

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- The interpretation $\mathcal{I}$ satisfies the GCI $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- The interpretation $\mathcal{I}$ is a model of the TBox $\mathcal{T}$ iff it satisfies all the GCIs in $\mathcal{T}$.

$$\mathsf{Talk} \sqcap \forall \mathsf{attends}^{-1}.\mathsf{Sleeping} \sqsubseteq \mathsf{Boring}$$

$$\mathsf{Author} \sqcap \mathsf{PCchair} \sqsubseteq \bot$$

2.2. Terminological knowledge

Definition 2.4 (GCIs and TBoxes)

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- A TBox is a finite set of GCIs.
- The interpretation $\mathcal{I}$ satisfies the GCI $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- The interpretation $\mathcal{I}$ is a model of the TBox $\mathcal{T}$ iff it satisfies all the GCIs in $\mathcal{T}$.

Notation: two TBoxes are called equivalent if they have the same models

More GCIs $\Longrightarrow$ less models

Lemma 2.5

If $\mathcal{T} \subseteq \mathcal{T}'$ for two TBoxes $\mathcal{T}, \mathcal{T}'$, then each model of $\mathcal{T}'$ is also a model of $\mathcal{T}$.

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Restricted TBoxes

concept definitions and acyclic TBoxes

Definition 2.6

A concept definition is of the form $A \equiv C$ where

- A is a concept name;
- C is a concept description.

The interpretation $\mathcal{I}$ satisfies the concept definition $A \equiv C$ iff $A^{\mathcal{I}} = C^{\mathcal{I}}$.

abbreviation for the two GCIs
$A \sqsubseteq C$ and $C \sqsubseteq A$

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Restricted TBoxes

concept definitions and acyclic TBoxes

Definition 2.6 (continued)

An acyclic TBox is a finite set of concept definitions that

- does not contain multiple definitions;
- does not contain cyclic definitions.

multiple definition

$$\cancel{A \equiv C}$$
$$\cancel{A \equiv D}$$

for $C \neq D$

$$\cancel{A \equiv B \sqcap \forall r.P}$$
$$\cancel{B \equiv P \sqcap \forall r.C}$$
$$\cancel{C \equiv \exists r.A}$$

cyclic definition

No cyclic definitions:
there is no sequence $A_1 \equiv C_1, \ldots A_n \equiv C_n \in \mathcal{T}$ $(n \geq 1)$ such that

- A_{i+1} occurs in C_i $(1 \leq i < n)$
- A_1 occurs in C_n

Case $n = 1$?

Restricted TBoxes

Definition 2.6 (continued)

An acyclic TBox is a finite set of concept definitions that

- does not contain multiple definitions;
- does not contain cyclic definitions.

The interpretation $\mathcal{I}$ is a model of the acyclic TBox $\mathcal{T}$ iff it satisfies all its concept definitions: $A^{\mathcal{I}} = C^{\mathcal{I}}$ for all $A \equiv C \in \mathcal{T}$

Given an acyclic TBox, we call a concept name A occurring in $\mathcal{T}$ a

- defined concept iff there is C such that $A \equiv C \in \mathcal{T}$;
- primitive concept otherwise.

Example

of an acyclic TBox

$$
\begin{array}{rcl}
\text{Woman} & \equiv & \text{Person} \sqcap \text{Female} \\
\text{Man} & \equiv & \text{Person} \sqcap \neg\text{Female} \\
\text{Talk} & \equiv & \exists \text{topic}.\top \\
\text{Speaker} & \equiv & \text{Person} \sqcap \exists \text{gives}.\text{Talk} \\
\text{Participant} & \equiv & \text{Person} \sqcap \exists \text{attends}.\text{Talk} \\
\text{BusySpeaker} & \equiv & \text{Speaker} \sqcap (\geq 3\, \text{gives}.\text{Talk}) \\
\text{BadSpeaker} & \equiv & \text{Speaker} \sqcap \forall \text{gives}.(\forall \text{attends}^{-1}.(\text{Bored} \sqcup \text{Sleeping}))
\end{array}
$$

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Acyclic TBoxes

Proposition 2.7

For every acyclic TBox $\mathcal{T}$ we can effectively construct an equivalent acyclic TBox $\widehat{\mathcal{T}}$ such that the right-hand sides of concept definitions in $\widehat{\mathcal{T}}$ contain only primitive concepts.

Proof: blackboard

Acyclic TBoxes

an important result

Proposition 2.7

For every acyclic TBox $\mathcal{T}$ we can effectively construct an equivalent acyclic TBox $\widehat{\mathcal{T}}$ such that the right-hand sides of concept definitions in $\widehat{\mathcal{T}}$ contain only primitive concepts.

We call $\widehat{\mathcal{T}}$ the *expanded version* of $\mathcal{T}$.

Acyclic TBoxes

an important result

Given an acyclic TBox $\mathcal{T}$, a primitive interpretation $\mathcal{J}$ for $\mathcal{T}$ consists of a nonempty set $\Delta^{\mathcal{J}}$ together with an extension mapping $\cdot^{\mathcal{J}}$, that maps

- primitive concepts P to sets $P^{\mathcal{J}} \subseteq \Delta^{\mathcal{J}}$
- role names r to binary relations $r^{\mathcal{J}} \subseteq \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}}$

The interpretation $\mathcal{I}$ is an extension of the primitive interpretation $\mathcal{J}$ iff $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and

- $P^{\mathcal{J}} = P^{\mathcal{I}}$ for all primitive concepts P
- $r^{\mathcal{J}} = r^{\mathcal{I}}$ for all role names r

Corollary 2.8

Let $\mathcal{T}$ be an acyclic TBox.
Any primitive interpretation $\mathcal{J}$ has a unique extension to a model of $\mathcal{T}$.

Proof: blackboard

Relationship with First-Order Logic

$\mathcal{ALC}$-TBoxes can be be translated into first-order logic:

$$\pi(\mathcal{T}) := \bigwedge_{C \sqsubseteq D \in \mathcal{T}} \forall x.(\pi_x(C) \rightarrow \pi_x(D))$$

Lemma 2.9

Let $\mathcal{T}$ be a TBox and $\tau(\mathcal{T})$ its translation into first-order logic.
Then $\mathcal{T}$ and $\pi(\mathcal{T})$ have the same models.

Proof: blackboard

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reasoning component

- derive implicitly respresented knowledge (e.g., subsumption)
- “practical” algorithms

2.3. Assertional knowledge

Definition 2.10 (Assertions and ABoxes)

An assertion is of the form

$a : C$ (concept assertion) or $(a, b) : r$ (role assertion)

where C is a concept description, r is a role, and a, b are individual names from a set $\mathbf{I}$ of such names (disjoint with $\mathbf{C}$ and $\mathbf{R}$).

An ABox is a finite set of assertions.

An interpretation $\mathcal{I}$ is a model of an ABox $\mathcal{A}$ if it satisfies all its assertions:

$$a^{\mathcal{I}} \in C^{\mathcal{I}} \quad \text{for all } a : C \in \mathcal{A}$$
$$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}} \quad \text{for all } (a, b) : r \in \mathcal{A}$$

$\mathcal{I}$ assigns elements $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ to individual names $a \in \mathbf{I}$

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2.3. Assertional knowledge

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An assertion is of the form

$a : C$ (concept assertion) or $(a, b) : r$ (role assertion)

where C is a concept description, r is a role, and a, b are individual names from a set **I** of such names (disjoint with **C** and **R**).

An ABox is a finite set of assertions.

$$\begin{array}{ll} \text{FRANZ} : \text{Lecturer}, & (\text{FRANZ}, \text{TU03}) : \text{teaches}, \\ \text{TU03} : \text{Tutorial}, & (\text{TU03}, \text{RinDL}) : \text{topic}, \\ \text{RinDL} : \text{DL} & \end{array}$$

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Relationship with First-Order Logic

$\mathcal{ALC}$-ABoxes can be be translated into first-order logic:

$$\pi(\mathcal{A}) := \bigwedge_{a\,:\,C \in \mathcal{T}} \pi_x(C)(a) \wedge \bigwedge_{(a,b)\,:\,r \in \mathcal{T}} r(a, b)$$

individual names are
viewed as constants

<u>Lemma 2.11</u>

Let $\mathcal{A}$ be an ABox and $\pi(\mathcal{A})$ its translation into first-order logic.
Then $\mathcal{A}$ and $\pi(\mathcal{A})$ have the same models.

Proof: easy

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Knowledge Bases

Definition 2.12

A knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$.

The interpretation $\mathcal{I}$ is a model of the knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ iff it is a model of $\mathcal{T}$ and a model of $\mathcal{A}$.

First-order translation: $\pi(\mathcal{K}) := \pi(\mathcal{T}) \wedge \pi(\mathcal{A})$

Lemma 2.13

Let $\mathcal{K}$ be a knowledge base and $\tau(\mathcal{K})$ its translation into first-order logic. Then $\mathcal{K}$ and $\pi(\mathcal{K})$ have the same models.

Proof: immediate consequence of Lemma 2.9 and Lemma 2.11

Additional constructors

Individual names can also be used as concept constructors to increase the expressive power of the concept description language.

They yield a singleton set consisting of the extension of the individual name.

Nominals

letter $\mathcal{O}$ in the naming scheme

Nominals: $\{a\}$ for $a \in \mathbf{I}$ with semantics

$$\{a\}^{\mathcal{I}} \quad := \quad \{a^{\mathcal{I}}\}$$

Nominals can be used to express ABox assertions using GCIs:

$$a : C \quad \text{is expressed by} \quad \{a\} \sqsubseteq C$$

$$(a, b) : r \quad \text{is expressed by} \quad \{a\} \sqsubseteq \exists r.\{b\}$$

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Description logic system

structure

2.1

description language

- constructors for building complex concepts out of atomic concepts and roles
- formal, logic-based semantics

TBox 2.2

defines the terminology of the application domain

ABox 2.3

states facts about a specific “world”

knowledge base

2.4

reasoning component

- derive implicitly respresented knowledge (e.g., subsumption)
- “practical” algorithms

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2.4. Reasoning Problems and Services

make implicitly represented knowledge explicit

Definition 2.14 (terminological reasoning)

Let $\mathcal{T}$ be a TBox.

Satisfiability:

C is satisfiable w.r.t. $\mathcal{T}$ iff $C^\mathcal{I} \neq \emptyset$ for some model $\mathcal{I}$ of $\mathcal{T}$.

Subsumption:

C is subsumed by D w.r.t. $\mathcal{T}$ ($C \sqsubseteq_\mathcal{T} D$) iff

$C^\mathcal{I} \subseteq D^\mathcal{I}$ for all models $\mathcal{I}$ of the TBox $\mathcal{T}$.

Equivalence:

C is equivalent to D w.r.t. $\mathcal{T}$ ($C \equiv_\mathcal{T} D$) iff

$C^\mathcal{I} = D^\mathcal{I}$ for all models $\mathcal{I}$ of the TBox $\mathcal{T}$.

Terminological Reasoning

Note:

If $\mathcal{T} = \emptyset$, then satisfiability/subsumption/equivalence w.r.t. $\mathcal{T}$ is simply called satisfiability/subsumption/equivalence and we write $\sqsubseteq$ and $\equiv$.

Examples:

- $A \sqcap \neg A$ and $\forall r.A \sqcap \exists r.\neg A$ are not satisfiable (unsatisfiable)
- $A \sqcap \neg A$ and $\forall r.A \sqcap \exists r.\neg A$ are equivalent
- $A \sqcap B$ is subsumed by A and by B.
- $\exists r.(A \sqcap B)$ is subsumed by $\exists r.A$ and by $\exists r.B$
- $\forall r.(A \sqcap B)$ is equivalent to $\forall r.A \sqcap \forall r.B$
- $\exists r.A \sqcap \forall r.B$ is subsumed by $\exists r.(A \sqcap B)$

Properties of Subsumption

Lemma 2.15

- The subsumption relation $\sqsubseteq_{\mathcal{T}}$ is a pre-order on concept descriptions, i.e.,
  - $C \sqsubseteq_{\mathcal{T}} C$ (reflexive)
  - $C \sqsubseteq_{\mathcal{T}} D \wedge D \sqsubseteq_{\mathcal{T}} E \to C \sqsubseteq_{\mathcal{T}} E$ (transitive)

  It is not a partial order since it is not antisymmetric:

  - $C \sqsubseteq_{\mathcal{T}} D \wedge D \sqsubseteq_{\mathcal{T}} C \not\to C = D$

- The constructors existential restriction and value restriction are monotonic w.r.t. subsumption, i.e.,
  - $C \sqsubseteq_{\mathcal{T}} D \to \exists r.C \sqsubseteq_{\mathcal{T}} \exists r.D \wedge \forall r.C \sqsubseteq_{\mathcal{T}} \forall r.D$

- Subsumption reasoning is monotonic, i.e., if $\mathcal{T} \subseteq \mathcal{T}'$, then
  - $C \sqsubseteq_{\mathcal{T}} D \to C \sqsubseteq_{\mathcal{T}'} D$

Proof: blackboard

Basic equivalences

Lemma 2.16

- $\neg\top \equiv \bot$
- $C \sqcup D \equiv \neg(\neg C \sqcap D)$
- $\forall r.C \equiv \neg\exists r.\neg C$
- $\exists r.\neg\bot \equiv \bot$
- $\neg(\geq n\, r.C) \equiv (\leq n-1\, r.C)$ if $n \geq 1$
- $(\geq 0\, r.C) \equiv \top$
- $(\leq 0\, r.C) \equiv \forall r.\neg C$

Proof: blackboard

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Assertional Reasoning

Definition 2.17 (assertional reasoning)

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base.

Consistency:

$\mathcal{K}$ is consistent iff there exists a model of $\mathcal{K}$.

Instance:

a is an instance of C w.r.t. $\mathcal{K}$ iff $a^\mathcal{I} \in C^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{K}$.

Lemma 2.18

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base.

If a is an instance of C w.r.t. $\mathcal{K}$ and $C \sqsubseteq_\mathcal{T} D$,

then a is an instance of D w.r.t. $\mathcal{K}$.

Proof: exercise

Reductions

between reasoning problems

There are the following polynomomial time reductions between the introduced reasoning problems:

$$
\begin{tikzcd}
\text{equivalence} \arrow[r, shift left, "1."] & \text{subsumption} \arrow[l, shift left, "2."] \arrow[r, shift left, "3."] & \text{satisfiability} \arrow[l, shift left, "4."] \arrow[d, "5."] \\
 & \text{instance} \arrow[r, shift left, "6."] & \text{consistency} \arrow[l, shift left, "7."]
\end{tikzcd}
$$

This holds not only for $\mathcal{ALC}$, but for all DLs that have the constructors conjunction and negation.

Theorem 2.19

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base, C, D concept descriptions, and $a \in \mathbf{I}$.

1. $C \equiv_{\mathcal{T}} D$ iff $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} C$

2. $C \sqsubseteq_{\mathcal{T}} D$ iff $C \equiv_{\mathcal{T}} C \sqcap D$

3. $C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is unsatisfiable w.r.t. $\mathcal{T}$

4. C is satisfiable w.r.t. $\mathcal{T}$ iff $C \not\sqsubseteq_{\mathcal{T}} \bot$

5. C is satisfiable w.r.t. $\mathcal{T}$ iff $(\mathcal{T}, \{a : C\})$ is consistent

6. a is an instance of C w.r.t. $\mathcal{K}$ iff $(\mathcal{T}, \mathcal{A} \cup \{a : \neg C\})$ is inconsistent

7. $\mathcal{K}$ is consistent iff a is not an instance of $\bot$ w.r.t. $\mathcal{K}$

Proof: blackboard

Reduction

getting rid of acyclic TBoxes

Expansion of concepts and ABoxes:

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base, where $\mathcal{T}$ is acyclic, and C a concept description.

The expanded versions $\widehat{C}$ and $\widehat{\mathcal{A}}$ of C and $\mathcal{A}$ w.r.t. $\mathcal{T}$ are obtained as follows:

- replace all defined concepts occurring in C and $\mathcal{A}$ by their definitions in the expanded version $\widehat{\mathcal{T}}$ of $\mathcal{T}$.

$\mathcal{T}$

$$\begin{array}{rcl} \text{Woman} & \equiv & \text{Person} \sqcap \text{Female} \\ \text{Talk} & \equiv & \exists \text{topic}.\top \\ \text{Speaker} & \equiv & \text{Person} \sqcap \exists \text{gives}.\text{Talk} \end{array}$$

$C = \text{Woman} \sqcap \text{Speaker}$ expands to

$$\widehat{C} = \text{Person} \sqcap \text{Female} \sqcap \text{Person} \sqcap \exists \text{gives}.(\exists \text{topic}.\top)$$

Reduction

getting rid of acyclic TBoxes

Expansion of concepts and ABoxes:

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base, where $\mathcal{T}$ is acyclic, and C a concept description.

The expanded versions $\widehat{C}$ and $\widehat{\mathcal{A}}$ of C and $\mathcal{A}$ w.r.t. $\mathcal{T}$ are obtained as follows:

- replace all defined concepts occurring in C and $\mathcal{A}$ by their definitions in the expanded version $\widehat{\mathcal{T}}$ of $\mathcal{T}$.

Proposition 2.20

1. C is satisfiable w.r.t. $\mathcal{T}$ iff $\widehat{C}$ is satisfiable
2. $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent iff $(\emptyset, \widehat{\mathcal{A}})$ is consistent

Proof: blackboard

Reduction

getting rid of acyclic TBoxes

Expansion of concepts and ABoxes:

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base, where $\mathcal{T}$ is acyclic, and C a concept description.

The expanded versions $\widehat{C}$ and $\widehat{\mathcal{A}}$ of C and $\mathcal{A}$ w.r.t. $\mathcal{T}$ are obtained as follows:

- replace all defined concepts occurring in C and $\mathcal{A}$ by their definitions in the expanded version $\widehat{\mathcal{T}}$ of $\mathcal{T}$.

Proposition 2.20

1. C is satisfiable w.r.t. $\mathcal{T}$ iff $\widehat{C}$ is satisfiable
2. $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent iff $(\emptyset, \widehat{\mathcal{A}})$ is consistent

Similar reductions exist for the other reasoning problems.

Reduction

getting rid of the TBox

This reduction is in general not polynomial,

since the expanded versions may be exponential in the size of $\mathcal{T}$.

$$
\begin{array}{rcl}
A_0 & \equiv & \forall r.A_1 \sqcap \forall s.A_1 \\
A_1 & \equiv & \forall r.A_2 \sqcap \forall s.A_2 \\
 & \vdots & \\
A_{n-1} & \equiv & \forall r.A_n \sqcap \forall s.A_n
\end{array}
$$

The size of $\mathcal{T}$ is linear in n,

but the expansion version $\widehat{A_0}$ of A_0 contains A_n 2^n times.

Proof: induction on n

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Relationship with First-Order Logic

Reasoning in $\mathcal{ALC}$ can be translated into reasoning in first-order logic:

Lemma 2.21

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base, C, D be $\mathcal{ALC}$-concept descriptions, and a an individual name.

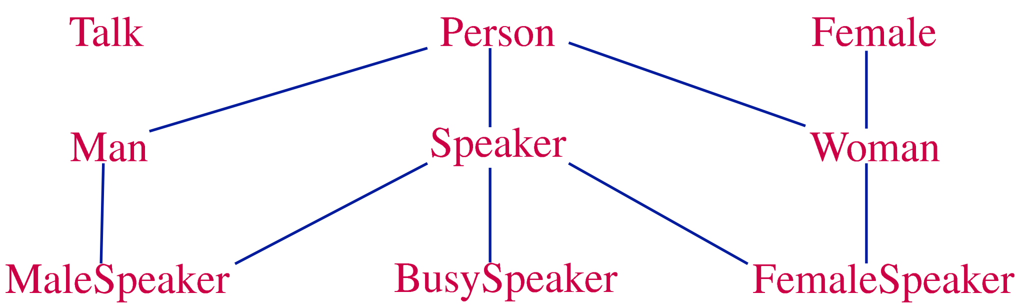
1. $C \sqsubseteq_{\mathcal{T}} D$ iff $\pi(\mathcal{T}) \models \forall x.(\pi_x(C)(x) \rightarrow \pi_x(D)(x))$
2. $\mathcal{K}$ is consistent iff $\pi(\mathcal{K})$ is consistent
3. a is an instance of C w.r.t. $\mathcal{K}$ iff $\pi(\mathcal{K}) \models \pi_x(C)(a)$

Proof: blackboard

Classification

Computing the subsumption hierarchy of all concept names occurring in the TBox.

$$
\begin{array}{rcl}
\mathsf{Man} & \equiv & \mathsf{Person} \sqcap \neg\mathsf{Female} \\
\mathsf{Woman} & \equiv & \mathsf{Person} \sqcap \mathsf{Female} \\
\mathsf{MaleSpeaker} & \equiv & \mathsf{Man} \sqcap \exists \mathsf{gives.Talk} \\
\mathsf{FemaleSpeaker} & \equiv & \mathsf{Woman} \sqcap \exists \mathsf{gives.Talk} \\
\mathsf{Speaker} & \equiv & \mathsf{FemaleSpeaker} \sqcup \mathsf{MaleSpeaker} \\
\mathsf{BusySpeaker} & \equiv & \mathsf{Speaker} \sqcap (\geq 3\, \mathsf{gives.Talks})
\end{array}
$$



Realization

Computing the most specific concept names in the TBox to which an ABox individual belongs.

$$
\begin{aligned}
\text{Man} &\equiv \text{Person} \sqcap \neg\text{Female} \\
\text{Woman} &\equiv \text{Person} \sqcap \text{Female} \\
\text{MaleSpeaker} &\equiv \text{Man} \sqcap \exists \text{gives}.\text{Talk} \\
\text{FemaleSpeaker} &\equiv \text{Woman} \sqcap \exists \text{gives}.\text{Talk} \\
\text{Speaker} &\equiv \text{FemaleSpeaker} \sqcup \text{MaleSpeaker} \\
\text{BusySpeaker} &\equiv \text{Speaker} \sqcap (\geq 3\,\text{gives}.\text{Talks})
\end{aligned}
$$

$\text{FRANZ} : \text{Man}, \quad (\text{FRANZ}, \text{T1}) : \text{gives},$
$\text{T1} : \text{Talk}$

FRANZ is an instance of Man, Speaker, $\underbrace{\text{MaleSpeaker}}_{\text{most specific}}$.