Section 3

Default Logic

Subsection 3.1

Introducing defaults and default logics

Introducing defaults and default logics: an example

Suppose you are asked how you get to the university in the morning.

By bike (usually)! $\dfrac{gotoWork : byBike}{byBike}$

new information: It is snowing heavily and your bike's tire is flat.
(You cannot assume that you go by bike.))

The default is no longer applicable

revise previous conclusion

Why can classical logic not model this?
$goToUni \wedge \neg snow \wedge \neg FlatTire \longrightarrow useBike$

- There are more reasons not to use the bike, e.g. broke brakes, demonstration in the city, . . . The formula would need to list all possible obstacles!
- All preconditions would need to be established to be true, so that the rule applies!

Default reasoning

Defaults can be used to model several forms of (common sense) reasoning.

- Prototypical reasoning: most instances of a category have a property.

  "Typically, children have parents" $\dfrac{child(X) : hasParents(X)}{hasParents(X)}$

- No-risk reasoning: concerns situations where a conclusion is drawn even if it is not the most probable, because another conclusion could lead to disaster.

  "in absence of evidence to the contrary assume the accused is innocent" $\dfrac{accused(X) : innocent(X)}{innocent(X)}$

- Best-guess reasoning: for instance, we know that there are some shopping centers in this city and some are open on Sundays, but we don't know which one. We would try the closest first, although we don't have evidence of it.

$$\frac{closest(X) : openSundays(X)}{openSundays(X)}$$

Default reasoning appears in many application domains: legal reasoning, diagnosis, reasoning about actions, etc.

Introducing defaults and default logics

- Default logics were introduced by Ray Reiter in 1980
- Default reasoning appears when reasoning is done under the closed world assumption and using inference rules that admit exceptions (rules that hold under the normality assumption)
  "…in absence of any information to the contrary, assume …"
- Classical inference rules sanction the derivation of a formula whenever some other formulas are derived.
- Default rules require an additional consistency condition to hold.

  E.g.: the rule "normally birds fly" is represented as $\dfrac{bird(x) : flies(x)}{flies(x)}$

  This states that:
  "if $bird(J)$ is given or derived for a particular ground term J and $flies(J)$ is consistent (there is no information that $\neg flies(J)$ holds), then $flies(J)$ can be derived "by default".

  Consistent with what? Set of formulas that can "reasonably" be accepted based on the available information.

Definition 3.1 (Default theory)

A default theory is a pair (W, D) consisting of

- W: a set of FOL formulas (called facts or axioms)
- D: a countable set of defaults

A default δ has the form

$$\frac{\varphi : \psi_1, \ldots, \psi_n}{\chi},$$

where $\varphi, \psi_1, \ldots, \psi_n$, and χ are closed FOL formulas and $n > 0$.

The formula

- φ is called the prerequisite (denoted by $pre(\delta)$),
- $\psi_1, \ldots, \psi_n$ the justifications (denoted by $just(\delta)$), and
- χ the consequent of δ (denoted by $cons(\delta)$).

Why closed formulas?

Actually,

$$\frac{bird(x)\ :\ flies(x)}{flies(x)}$$

is not a default according to Definition 3.1. We call such “defaults” open defaults. An open default is interpreted as a default schema representing a (possibly infinite) set of defaults.

A default schema differs from a default in that $\varphi, \psi_1, \ldots, \psi_n, \chi$ are arbitrary FOL formulas (may contain free variables). A default schema defines a set of defaults

$$\frac{\varphi\sigma\ :\ \psi_1\sigma, \ldots, \psi_n\sigma}{\chi\sigma}$$

for all ground substitutions σ that assign values to all free variables occurring in the schema.

$\leadsto$ Free variables : interpreted as being universally quantified over the whole default schema

Why closed formulas?

The open default

$$\frac{\mathit{bird}(x) \ : \ \mathit{flies}(x)}{\mathit{flies}(x)}$$

would read under

- universally quantified variables as
  "If all X are birds, and if for all X we may assume that they fly, then we conclude that all X fly."
  - Does not match the intuition.
  - Would only be applicable, if every object in the domain is a bird.
- existentially quantified variables as
  "If there is a bird and if there is an X that flies, then conclude that there is some flying object."
  - Would not allow to conclude from $\mathit{bird}(\mathit{tweety})$ that $\mathit{flies}(\mathit{tweety})$ holds.
  - instead we would conclude: $\exists X \ \mathit{flies}(X)$

Towards the semantics of defaults

The informal meaning of a default $\frac{\varphi : \psi_1, \ldots, \psi_n}{\chi}$ is:

"If φ' is known, and if it is consistent to assume that $\psi_1, \ldots, \psi_n$, then conclude χ."

In the formal semantics we must say

1. where φ should be included
2. with what should $\psi_1, \ldots, \psi_n$ be consistent

With what should $\psi_1, \ldots, \psi_n$ be consistent? A first attempt: the facts.
Consider the default

$$\frac{\mathit{friend}(X, Y) \land \mathit{friend}(Y, Z) \ : \ \mathit{friend}(X, Z)}{\mathit{friend}(X, Z)}$$

Given the facts: $\mathit{friend}(\mathit{tom}, \mathit{bob})$, $\mathit{friend}(\mathit{bob}, \mathit{sally})$, $\mathit{friend}(\mathit{sally}, \mathit{tina})$.
Wanted conclusion: $\mathit{friend}(\mathit{tom}, \mathit{tina})$

This is only possible if we apply the appropriate instance of the default schema to $\mathit{friend}(\mathit{sally}, \mathit{tina})$ and $\mathit{friend}(\mathit{tom}, \mathit{sally})$. But $\mathit{friend}(\mathit{tom}, \mathit{sally})$ is derived by a previous application of the default schema!
Without this intermediate result and from the facts alone, we could not derive this.

Example 3.2

Let's consider $T = (W, D)$ with $W = \{green, ADACmember\}$ and $D = \{\delta_1, \delta_2\}$, where

$$\delta_1 = \frac{green\ :\ \neg likesCars}{\neg likesCars} \quad \text{and} \quad \delta_2 = \frac{ADACmember\ :\ likesCars}{likesCars}$$

If consistency is tested against W, both defaults can be applied. Deriving $\neg$*likesCars* and *likesCars*, which is a contradiction!

Alternative:
apply the first default δ_1, check for consistency with the knowledge derived so far.
Would block the application of the second default δ_2.

Informal semantics of defaults

A general formulation:
If φ is currently known, and if all ψ_i are consistent with the current knowledge base, then conclude χ.
The current knowledge base E is obtained from

- the facts and
- the consequents of some defaults that have been applied previously.

A more formal version:
$\delta = \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi}$ is applicable to a deductively closed set of formulas E iff $\varphi \in E$ and $\neg\psi_1 \notin E, \ldots, \neg\psi_1 \notin E$.

Towards extensions

Example 3.2 suggests that there can be several competing knowledge bases which maybe inconsistent with each other.

Extensions
represent possible world views which are based on the given default theories. They seek to extend the set of known facts with "reasonable" conjectures based on the available defaults.

Desirable properties of extensions:

- an extension should include the set W of facts—the certain information
- an extension should be deductively closed
  (Keep classical reasoning! Derive more from the defaults)
- an extension should be closed under the application of the defaults. Apply defaults exhaustively.

  Formally: if $\frac{\varphi : \psi_1, \ldots, \psi_n}{\chi} \in D$, $\varphi \in E$ and $\neg\psi_1 \notin E, \ldots, \neg\psi_n \notin E$ then $\chi \in E$.

Extensions are maximal possible world views.

Towards extensions – unwanted effects

1. "Ungrounded" beliefs
An extension must not contain "ungrounded" beliefs, i.e. every formula in the extension must be derivable from W and the consequents of applied defaults.
We require extensions to be minimal w.r.t. to these properties.

Consider: $T = (W, D)$ with $W = \{german\}$ and $D = \left\{ \frac{german : drinksBeer}{drinksBeer} \right\}$
Now, $E = Th(\{german, \neg drinksBeer\})$ is minimal w.r.t. to the properties, but unintuitive.

2. Applications of defaults can contradict the application of an earlier default.
Consider:

$$\frac{true \ : \ creditworthy}{approveCredit}, \quad \frac{true \ : \ \neg creditworthy}{\neg creditworthy}$$

We apply the first default, since nothing contradicts the assumption *creditworthy*.
We then apply the second, since $\neg$*creditworthy* is consistent with the knowledge, $\neg$*creditworthy* is derived.
Inclusion of $\neg$*creditworthy* shows a posteriori that, we should not have assumed *creditworthy*.

Subsection 3.2

Operational semantics of Default Logic

- based on the process in which inferences are drawn
- gives a procedure that can be applied

Idea:

- apply defaults as long as possible
- If a default should not have been applied, backtrack and try an alternative

Operational Semantics

Given a default theory $T = (W, D)$ let $\Pi = (\delta_0, \delta_1 \ldots)$ be (a finite or infinite) sequence of defaults from D without multiple occurrences.
(Possible order in which some defaults from D are applied.)

$\Pi[k]$ denotes the initial segment of sequence Π of length k.[3]
Each sequence Π is associated with two sets: $In(\Pi)$ and $Out(\Pi)$

- $In(\Pi) = Th\big(W \cup \{cons(\delta) \mid \delta \text{ occurs in } \Pi\}\big)$.
- $Out(\Pi) = \{\neg\psi \mid \psi \in just(\delta) \text{ for some } \delta \text{ in } \Pi\}$.

Intuition:

- $In(\Pi)$ represents the current knowledge base after the defaults in Π have been applied
- $Out(\Pi)$ represents the formulas that should not become true even after subsequent application of other defaults.

[3] We assume (from now on) that the length of Π is at least k.

Example 3.3

Consider $T = (W, D)$ with $W = \{a\}$ and the defaults from D:

$$\delta_1 = \frac{a : \neg b}{\neg b}, \quad \delta_2 = \frac{b : c}{c}$$

For $\Pi_a = (\delta_1)$ we have $In(\Pi_a) = Th(\{a, \neg b\})$ and $Out(\Pi_a) = \{b\}$.
For $\Pi_b = (\delta_2, \delta_1)$ we have $In(\Pi_b) = Th(\{a, c, \neg b\})$ and $Out(\Pi_b) = \{\neg c, b\}$

We have not assured that the defaults can be applied in the order given.
(δ_2, δ_1) cannot be applied in this order, since $b \notin In(()) = Th(W) = Th(a)$.

"Applicable sequences" are formalized by the notion of a process.

Definition 3.4 (Process, successful, closed)

Π is a process of T iff δ_k is applicable to $In(\Pi[k])$ for every k s.t.[4] δ_k occurs in Π.
Let Π be a process. We define:

- Π is successful iff $In(\Pi) \cap Out(\Pi) = \emptyset$. Otherwise, it is failed.
- Π is closed iff every $\delta \in D$ that is applicable to $In(\Pi)$ already occurs in $In(\Pi)$

Intuition:
Success of a process captures that is was "okay" to have assumed the justifications of the applied defaults; no formula $\neg\psi \in Out(\Pi)$ is part of the current knowledge base, so it was consistent to assume ψ.

Closed processes correspond to the extension being closed under application of the defaults.

[4] "such that"

Example: properties of processes

Consider the default theory $T = (W, D)$ with $W = \{a\}$ and D containing

$$\delta_1 = \frac{a : \neg b}{d}, \quad \delta_2 = \frac{true : c}{b}$$

$\Pi_1 = (\delta_1)$
is successful, but not closed, since δ_2 may be applied to $In(\Pi_1) = Th(\{a, d\})$.

$\Pi_2 = (\delta_2, \delta_1)$
is closed, but not successful. Since both $In(\Pi_2) = Th(a, b, d)$ and $Out(\Pi_2) = \{b, \neg c\}$ contain b.

$\Pi_2 = (\delta_2)$
is a closed and successful process of T.

Definition 3.5 (Extension)

Let T be a default theory. A set of formulas E is an extension of T iff there is some closed and successful process Π s.t. $E = Th(In(\Pi))$.

This definition may be applied directly to concrete examples.

To find a successful process, it suffices to generate a process Π, test whether $In(\Pi) \cap Out(\Pi) = \emptyset$ holds. If not, then backtrack.

A (in)finite default theory is a default theory, where D has (in)finitely many elements.

For finite default theories ensuring closure is conceptually easy: apply an applicable default that has not been applied yet, until no more a left.

How about closure of infinite default theories?

Closure of infinite theories

Lemma 3.6

An infinite process Π of a default theory $T = (W, D)$ is closed iff each default in D that is applicable to $In(\Pi[k])$, for infinitely many numbers k, is already contained in Π.

Proof: blackboard

A strategy that guarantees the closure of an infinite process Π must take care that any default which from k on, demands application, will eventually be applied.
This is the fairness condition from concurrent programming.

A systematic view on closed and successful processes

The process of finding an closed and successful process can be represented by a kind of (search) tree.

Definition 3.7 (Process tree)

Let $T = (W, D)$ be a default theory. A process tree is tree $G = (V, E)$, s.t. all nodes $v \in V$ are labeled with two sets of formulas:

- an In-set $In(v)$ and
- an Out-set $Out(v)$.

The root of G is labeled with $Th(W)$ as In-set and $\emptyset$ as Out-set.

The paths of a process are the paths in G starting at the root. A node v is expanded if $In(v) \cap Out(v) = \emptyset$, otherwise it is marked "failed" and is a leaf of the process tree.

If v is expanded it possesses for each default $\delta = \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi}$ one successor node w that

- does not appear on the path from the root node to v and
- is applicable to $In(v)$.
- is connected to v by a edge labeled with δ.
- is labeled with $Th(In(v) \cup \{\chi\})$ and $Out(v) \cup \{\neg\psi_1, \ldots, \neg\psi_n\}$.

Subsection 3.3

Original semantics of default logics

- original definition by Ray Reiter
- fixed point based, not constructive

Consistency w.r.t. to what?

When applying defaults we need to ensure consistency. But consistency w.r.t. to which theory?

We consider again Example 3.2:
$T = (W, D)$ with $W = \{green, ADACmember\}$ and $D = \{\delta_1, \delta_2\}$, where

$$\delta_1 = \frac{green\ :\ \neg likesCars}{\neg likesCars} \quad \text{and} \quad \delta_2 = \frac{ADACmember\ :\ likesCars}{likesCars}$$

Consistency w.r.t. to alone W is not enough.
Solution by Reiter: Use a theory that plays the role of a context or belief set. Check consistency against this context.

A formalization of this idea:
A default $\delta = \frac{\varphi\,:\,\psi_1,\ldots,\psi_n}{\chi}$ is applicable to a deductively closed set of formulas F w.r.t. belief set E (the context) iff $\varphi \in F$ and $\neg\psi_1 \notin E, \ldots, \neg\psi_n \notin E$ (each ψ_i is consistent with E).
Note that the concept "δ is applicable to E" is so far a special case where $E = F$.

Which context to use?

Observation:
If a default has been applied to a belief set E, a formula has been derived and is part of the knowledge base. Therefore it should be believed, i.e. become an element of belief set E.
On the other hand, E should contain only formulas that can be derived from the axioms in W by default application.

Definition 3.8 (Closure under a set of defaults w.r.t. a belief set)

Let D be a set of defaults and F a deductively closed set of formulas F.
F is closed under D w.r.t. belief set E iff, for every default $\delta \in D$ that is applicable to F w.r.t. belief set E, its consequent χ is also contained in E.

Lemma 3.9

Let $E' \subseteq E$ and F be a set of formulas closed under some set of defaults D w.r.t. E'. Then F is closed under D w.r.t. E.

Proof: exercise

Definition 3.10 ($\Lambda_T(E)$, extension)

Given $T = (W, D)$ and a set of formulas E. Let $\Lambda_T(E)$ be the smallest[5] set of formulas that is

- closed under deduction, i.e. contains all conclusions
- closed under D w.r.t. E.

E is an extension of T, iff $E = \Lambda_T(E)$.

Intuition:

- $\Lambda_T(E)$ contains all formulas that are sanctioned by T w.r.t. E.
- E is an extension, iff by the use of E as a belief set, exactly the formulas in E will be obtained from default application.

Observe: one first needs to guess E and then check whether the fixed-point equation is fulfilled.

[5] i.e. has smallest number of elements

Are the two definitions equivalent?

Theorem 3.11

Let $T = (W, D)$ be a default theory.
E is an extension of T (according Definition 3.5) iff $E = \Lambda_T(E)$.

Proof: blackboard

Minimality of Reiter's extensions

Reiter's characterization fulfills the desirable properties of an extension:

- include the set W of facts: E includes W.
- deductively closed: E is deductively closed
- closed under the application of the defaults: E is closed under D w.r.t. E

Claim: E is minimal w.r.t. these properties.
If E' is an extension and $E' \subseteq E$, then E' is closed under D w.r.t. E (by Lemma 3.9).
By definition, $E = \Lambda_T(E) \subseteq E'$ and thus $E' = E$.

Corollary 3.12 (Minimality of extension)

If, for two extensions E and E' of a default theory T, $E' \subseteq E$, then $E' = E$.

Theorem 3.13 (Consistency preservation)

A default theory $T = (W, D)$ has an inconsistent extension iff W is inconsistent.

Proof: Exercise

Corollary 3.14

If a default has an inconsistent extension E, then it is its only extension.

Theorem 3.15

Let $T = (W, D)$ be a default theory s.t. the set $M = W \cup \{\psi_1 \wedge \cdots \wedge \psi_n \wedge \chi \mid \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi}$ is a default in $D\}$ is consistent. Then T has exactly one extension.

Proof: blackboard

Nonmonotonic behavior may appear when the default theory is changed!

Example 3.16 (Changing the defaults)

Let $T_{ex}(W, D)$ be a default theory with $W = \emptyset$ and $D = \{\frac{true : a}{a}\}$.
T_{ex} has exactly one extension: $E = Th(\{a\})$.

- $\delta_1 = \frac{true : b}{\neg b}$. Then $(W, D \cup \{\delta_1\})$ has no extensions.
- $\delta_2 = \frac{b : c}{c}$. Then $(W, D \cup \{\delta_2\})$ has E as only extension.
- $\delta_3 = \frac{true : \neg a}{\neg a}$. Then $(W, D \cup \{\delta_3\})$ has two extensions: E and $Th(\{\neg a\})$.
- $\delta_4 = \frac{a : b}{b}$. Then $(W, D \cup \{\delta_4\})$ has the two extensions: $Th(\{a, b\})$.

Nonmonotonic behavior may appear when the default theory is changed!

Example 3.17 (Changing the facts)

Let $T_{ex}(W, D)$ be a default theory with $W = \emptyset$ and $D = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ with

$$\delta_1 = \frac{true\ :\ a, \neg c}{a},\ \delta_2 = \frac{a\ :\ b, \neg c}{b},\ \delta_3 = \frac{true\ :\ \neg a, c}{c},\ \delta_4 = \frac{d\ :\ e}{e},\ \delta_5 = \frac{f\ :\ g}{\neg g}$$

T_{ex} has two extensions: $E_1 = Th(\{a, b\})$ and $E_2 = Th(\{c\})$. Consider

- $W_1 = \{f\}$. (W_1, D) has no extensions.
- $W_2 = \{\neg a\}$. (W_2, D) has the only extension $Th(\{\neg a, c\})$.
- $W_3 = \{\neg a, \neg c, d\}$. (W_3, D) the only new extension: $Th(\{\neg a, \neg c, d, e\})$.
- $W_4 = \{d\}$. (W_4, D) the two extension: $Th(E_1 \cup \{d, e\})$ and $Th(E_2 \cup \{d, e\})$.

Subsection 3.4

Normal default logics

Definition 3.18 (Normal defaults, normal default theory)

A default is normal iff its consequent is its only justification. They have the form

$$\frac{\varphi : \psi}{\psi}.$$

A default theory $T = (W, D)$ is normal iff all defaults in D are normal.

Normal defaults

- have always extensions
- rule out "pathological cases" such as: $\frac{\mathit{true} : a}{\neg a}$
- have limited expressivity: no interactions among defaults

A normal default theory

draws the conclusion ψ when φ is known and it is consistent to conclude ψ.

Lemma 3.19

Each process of a normal default theory is successful.

Proof: blackboard

Expanding Π in a fair way, a closed and successful process is obtained. Thus we have established:

Theorem 3.20 (Existence of extensions)

Normal default theories always possess extensions. Every finite process Π *may be expanded to a closed process* Π'.

Default theories are not semi-monotonic

Semi-monotonicity

A class of default theories, is semi-monotonic iff the addition of default rules never eliminates, but extends or adds, new extensions.

Consider the general default theories:

$$T_1 = (\emptyset, D_1) \text{ with } D_1 = \left\{ \frac{true \;:\; \neg C}{D} \right\} \text{ and}$$

$$T_2 = (\emptyset, D_2) \text{ with } D_2 = \left\{ \frac{true \;:\; B}{C}, \frac{true \;:\; \neg C}{D} \right\}$$

The theory T_1 has one extension $E_1 = Th(\{D\})$.
However, the only extension of T_2 is $E_2 = Th(\{C\})$.

E_1 fails to be an extension of T_2 since B is consistent with E_2, hence $\frac{true : B}{C}$ is applicable and eliminating E_2 as possible extension.

Since we have $D_1 \subseteq D_2$, but $E_1 \not\subseteq E_2$, default logic is not semi-monotonic.

Properties of normal default theories

Theorem 3.21 (Semi-monotonicity)

Let $T = (W, D)$ and $T' = (W, D')$ be normal default theories s.t. $D \subseteq D'$. Then each extension of T is contained in an extension of T'.

Proof: blackboard

Theorem 3.22 (Orthogonality of extensions)

Let E and F be different extensions of a normal default theory T. Then $E \cup F$ is inconsistent.

Proof: blackboard

Limitations of normal default theories

Are normal default theories expressive enough to model common sense reasoning?

Statements such as:

- "Typically birds fly"
- "Assume the accused is innocent unless you know otherwise"

can be captured by normal defaults:

$$\frac{bird(x) \ : \ flies(x)}{flies(x)} \qquad \frac{accused(x) \ : \ innocent(x)}{innocent(x)}$$

Often a default rule on its own is normal, but problems arise when several defaults have to interact in a theory.

Limitations of normal default theories—example

Consider the example of a normal default theory:

$$T = \left(\{dropout(bill)\}, \left\{ \frac{dropout(x) \ : \ adult(x)}{adult(x)}, \frac{adult(x) \ : \ employed(x)}{employed(x)} \right\} \right)$$

T has the single extension $Th(\{dropout(bill), adult(bill), employed(bill)\})$. But it is counterintuitive to assume that Bill is employed!

How to prevent the application of the 2. default, if X is a dropout?

$$\frac{adult(x) \ : \ employed(x) \wedge \neg dropout(x)}{employed(x)}$$

But this is no longer a normal default!

Semi-normal Defaults

A default is a semi-normal default, if it has the form $\frac{\phi : \psi \wedge \chi}{\psi}$.
A default theory $T = (W, D)$ is semi-normal, if all defaults in D are semi-normal.

Do semi-normal default theories always have extensions? No.

Consider the example: $T = (W, D)$, with $W = \emptyset$ and

$$D = \left\{ \frac{true \ : \ \neg q \wedge p}{p}, \frac{true \ : \ \neg r \wedge q}{q}, \frac{true \ : \ \neg p \wedge r}{r} \right\}$$

Only some classes of restricted semi-normal theories do always have extensions.

Semi-normal Default Theories

Semi-normal default theories do not have ...

- Semi-monotonicity

  Consider:
  $T = (W, D)$ with $W = \emptyset$ and $D = \left\{ \frac{true : \neg q \wedge p}{p} \right\}$ and
  $T' = (W, D')$ with $W = \emptyset$ and $D' = \left\{ \frac{true : \neg q \wedge p}{p}, \frac{true : \neg r \wedge q}{q} \right\}$.

  We have $E = Th(\{p\})$ and $E = Th(\{q\})$ as the extensions of the theories, but $E \not\subseteq E'$.

- Success of all processes

  T' has a failed process.

- Orthogonality of extensions

  Consider:
  $T'' = (W, D)$ with $W = \emptyset$ and $D = \left\{ \frac{true : p \wedge q}{q}, \frac{true : \neg q \wedge \neg p}{\neg p} \right\}$.

  T'' has two extensions: $E_1 = Th(\{q\})$ and $E_2 = Th(\{\neg p\})$, but $E_1 \cup E_2$ is consistent.

Reasoning in Default Logics

Classical reasoning problems of default theories are:

- deciding whether a default theory has an extension
- deciding whether a given formula is element of all extensions.
  ("cautious reasoning" or "skeptical reasoning")
- deciding whether a given formula is element of one extension.
  ("brave reasoning" or "credulous reasoning")
  choosing a different extension, may yield different consequences.

Example 3.23 (Reprise of Example 3.17)

Let $T_{ex} = (W, D)$ be a default theory with $W = \{d\}$ and $D = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ with

$$\delta_1 = \frac{true\ :\ a, \neg c}{a},\ \delta_2 = \frac{a\ :\ b, \neg c}{b},\ \delta_3 = \frac{true\ :\ \neg a, c}{c},\ \delta_4 = \frac{d\ :\ e}{e},\ \delta_5 = \frac{f\ :\ g}{\neg g}$$

T_{ex} has two extensions: $E_1 = Th(\{d, e, a, b\})$ and $E_2 = Th(\{d, e, c\})$.

Formula $(d \wedge e)$ is a consequence for T_{ex} under cautious reasoning.

Formula c is a consequence for T_{ex} under brave reasoning.

Recap on complexity classes

Complexity class with oracle admits the use of a subroutine “at no cost”.

Polynomial hierarchy: the classes Π_k^P, Σ_k^P and Δ_k^P are defined as follows:

$$P = \Sigma_0^P = \Pi_0^P = \Delta_0^P$$

and for all $k \geq 0$:

$$\Sigma_{k+1}^P = NP^{\Sigma_k^P} \qquad \Pi_{k+1}^P = \text{co-}\Sigma_{k+1}^P \qquad \Delta_{k+1}^P = P^{\Sigma_k^P}$$

Note: $\Sigma_1^p = NP$, $\Pi_1^P = \text{co-}NP$ and $\Delta_{k+1}^P = P$.

Complexity of reasoning in default theories

Typically, default reasoning is harder than classical reasoning.

Computability / complexity results for different classes of default theories:

- FOL default theories: undecidable
  (since classical reasoning is already undecidable)

  Reasoning in default theories is not even semi-decidable, since computing an extension requires FOL consistency tests which are semi-decidable.
- normal default theories: undecidable
- propositional default theories:
  - deciding existence of an extension: Σ_2^P-complete
  - brave reasoning is Σ_2^P-complete
  - cautious reasoning is Π_2^P-complete