Section 6

Nonmonotonic Inference Relations

Subsection 6.1

Inference relations

Introduction

We have discussed three formalisms that provide nonmonotonic reasoning:
Default logics, Autoepistemic logic, and Circumscription.
They provide useful inference relations.

In this chapter we take a more abstract view:

- What are the properties of nonmonotonic inference relations in general—independent of a particular formalism?
- How to compare the different approaches?

Inferences in the discussed formalisms:

- Circumscription uses minimal, i.e., preferred models, yielding a preferential inference relation.
- Default logic and autoepistemic logic use some kind of fixpoint construction.

What is the semantics of their inference relation?

Inference relations

Given knowledge about the world, when could a formula φ be reasonably concluded from a set of formulas M?

Let T be a set of first order sentences (the agent's knowledge).

- classical entailment
  The agent classically infers a formula φ, if φ holds in all models of T in which M holds.

But an agent's knowledge might be incomplete, so nonmonotonic (/defeasible) inference relation are interesting.

- under circumscription,
  the agent might infer φ from M if φ holds in every minimal model of T
- in default logic
  (if the agent's knowledge is given also by a set of default rules D), the agent might infer φ from M, if φ is in every extension of the default theory $(D, \{M\})$.

Inference relations can be modeled as binary relations on (sets of formulas of) a logic $\mathcal{L}$. We denote nonmonotonic inference relations by $\mid\!\sim$.

Which binary relations on $\mathcal{L}$ are (non-monotonic) inference relations?

For example: inference relation for default logic

In default logic the inference relation could be defined as:
Given a countable set of defaults D.

$$W \mid\!\sim \varphi \text{ iff } \varphi \text{ is included in all extensions of } (W, D).$$

or alternatively

$$W \mid\!\sim \varphi \text{ iff } \varphi \text{ is included in some extension of } (W, D).$$

The setting considered

In this chapter we consider:

- propositional logic
- inference relation $\mid\!\sim$
- the inference operation C defined (for a given set of formulas M) as:

$$C(M) = \{\varphi \mid M \mid\!\sim \varphi\}.$$

We will use the inference relation $\mid\!\sim$ and the inference operation $C(M)$ interchangeably. E.g.: '$M \mid\!\sim \varphi$ if $\varphi \in M$ can be formulated as '$M \subseteq C(M)$'.

The properties discussed in the following also hold for the classical inference relation $\vdash$.
(But not all properties of $\vdash$ hold for nonmonotonic reasoning.)

Subsection 6.2

Basic properties: pure conditions

Pure conditions

Consider the following properties of an inference operation C:

- $M \subseteq C(M)$ — Inclusion
- $C(M) = C(C(M))$ — Idempotence
- $M \subseteq N \subseteq C(M)$ implies $C(N) \subseteq C(M)$ — Cut
- $M \subseteq N \subseteq C(M)$ implies $C(M) \subseteq C(N)$ — Cautious Monotony
- $M \subseteq N \subseteq C(M)$ implies $C(M) = C(N)$ — Cumulativity
- $M \subseteq N$ implies $C(M) \subseteq C(N)$ — Monotony

These are called pure conditions, since they do not refer to any features of the underlying logic.

Intuition of the pure conditions

- Inclusion
  requires that the inference operation extends the set of formulas
- Idempotence
  requires that, after having applied the inference operation, another application does not add new formulas.
- Cut
  ensures that if the information in M is expanded by some proposition included in the closure $C(M)$, then no new conclusions are obtained.
- Cautious Monotony (converse of Cut)
  the addition of a lemma does not decrease the set of conclusions.
- Cumulativity (Cut and Cautious Monotony combined)
  ensures that lemmas can be safely used without affecting the supported conclusions.
- Monotony
  an extended set of premises gives an extended set of conclusions.

An inference relation that satisfies Inclusion, Idempotence, Cut, Cautious Monotony and Cumulativity is called a cumulative inference relation.

What is the minimal set of conditions for an inference relation to be cumulative?

Theorem 6.1

1. *Cut and Inclusion imply Idempotence.*
2. *Cautious monotony and Idempotence imply Cut.*

Proof: blackboard

Since Cumulativity implies Cautious Monotony and also Cut, we can conclude that Cumulativity and Inclusion are sufficient to obtain a cumulative inference relation.

Subsection 6.3

Basic properties: interaction with logical connectives

Properties linking classical and nonmonotonic logic

In the following we refer to propositional logic.

- $Th(M) \subseteq C(M)$ Supraclassicality
- $Th(C(M)) = C(M)$ Left Absorption
- $C(Th(M)) = C(M)$ Right Absorption
- $Th(C(M)) = C(M) = C(Th(M))$ Full Absorption

Theorem 6.2

Let C be a supraclassical inference relation.

1. *If C satisfies Idempotence, then it satisfies Left Absorption.*
2. *If C is Cumulative, then it satisfies Full Absorption.*

Proof: blackboard

Absorption properties establish some more properties

The following properties follow from Absorption and are linking nonmontonic inference and logical connectives.

- $M \mid\!\sim \varphi$ and $M \mid\!\sim \psi$ implies $M \mid\!\sim \varphi \wedge \psi$ — Right And
- $M \mid\!\sim \varphi$ and $\{\varphi\} \vdash \psi$ implies $M \mid\!\sim \psi$ — Right Weakening
- $M \mid\!\sim \varphi$ and $Th(M) = Th(N)$ implies $N \mid\!\sim \varphi$ — Left Logical Equivalence

A note on Right Weakening:
If $\mid\!\sim$ would have been used instead of $\vdash$, then a form of transitivity would have been obtained. Together with Supraclassicality this yields a form of monotony.
If Absorption, but not Monotony should hold, then transitivity must be given up.

Theorem 6.3

Let $\mid\!\sim$ be transitive and supraclassical. Then $\{\varphi\} \mid\!\sim \chi$ implies $\{\varphi \wedge \psi\} \mid\!\sim \chi$.

Proof: blackboard

Distribution

Another important property of inference relations:

- $C(M) \cap C(N) \subseteq C\big(Th(M) \cap Th(N)\big)$ Distribution

An inference relation that satisfies Distributivity is called distributive.

Lemma 6.4

If C satisfies Absorption, then Distribution is equivalent to the following properties:

1. *If $M = Th(M)$ and $N = Th(N)$ then $C(M) \cap C(N) \subseteq C(M \cap N)$*
2. $C(T \cup M) \cap C(T \cup N) \subseteq C\big(T \cup (Th(M) \cap Th(N))\big)$.

Proof: exercise

Properties that follow from Distribution

- $C(M \cup \{\varphi\}) \cap C(M \cup \{\psi\}) \subseteq C(M \cup \{\varphi \vee \psi\})$ Left Or
- $C(M \cup \{\varphi\}) \cap C(M \cup \{\neg\varphi\}) \subseteq C(M)$ Proof by Cases

Theorem 6.5

If C satisfies Distribution, Supraclassicality and Absorption, then C satisfies Left Or and Proof by Cases.

Proof: blackboard

Subsection 6.4

Properties of inference relations in default logic

Definition 6.6 (Inference relations in default logic)

Let D be a countable set of defaults.

- Skeptical inference relation $\vdash_{D,Ske}$ is defined as the intersection of all extensions of the default theory (M, D).
- Credulous inference relation $\vdash_{D,Cre}$ is defined as the union of all extensions of the default theory (M, D).

Credulous inference relation $\vdash_{D,Cre}$:

- tends to be irregular and violates most properties
- satisfies Left Logical Equivalence ($M \mid\!\sim \varphi$ and $Th(M) = Th(N)$ implies $N \mid\!\sim \varphi$)
- satisfies Right Weakening ($M \mid\!\sim \varphi$ and $\{\varphi\} \vdash \psi$ implies $M \mid\!\sim \psi$).

We concentrate in the following on skeptical inference relation $\vdash_{D,Ske}$.

A positive result for skeptical inference

Theorem 6.7

The skeptical inference relation $\vdash_{D,Ske}$ of default logic satisfies Cut and Absorption.

Proof: blackboard

Skeptical inference violates Cumulativity

Skeptical inference relation $\vdash_{D,Ske}$ does satisfy Cut, but it does not satisfy Cumulativity.

Example 6.8

To see that '$M \subseteq N \subseteq C(M)$ implies $C(M) = C(N)$' does not need to hold, consider $T = (W, D)$ with $W = \emptyset$ and

$$D = \left\{ \frac{true \ : \ a}{a}, \frac{a \vee b \ : \ \neg a}{\neg a} \right\}.$$

The only extension of T is $Th(\{a\})$.
Obviously, a is included in all extensions of T, i.e., $W \vdash_{D,Ske} a$.
From $(a \vee b) \in Th(\{a\})$, we get $W \vdash_{D,Ske} (a \vee b)$.

If we use $W' = W \cup \{a \vee b\}$, then the default theory $T' = (W', D)$ has two extensions: $Th(\{a\})$ and $Th(\{\neg a, b\})$. Thus $W \cup \{a \vee b\} \not\vdash_{D,Ske} a$.

Skeptical inference violates Distributivity

Skeptical inference relation $\vdash_{D,Ske}$ does not satisfy Distributivity.

Example 6.9

To see that '$C(M) \cap C(N) \subseteq C\big(Th(M) \cap Th(N)\big)$' does not need to hold, consider

$$D = \left\{ \frac{\varphi : \chi}{\chi}, \frac{\neg\varphi : \chi}{\chi} \right\}.$$

Then $\{\varphi\} \vdash_{D,Ske} \chi$ and $\{\neg\varphi\} \vdash_{D,Ske} \chi$, but $\big(Th(\{\varphi\}) \cap Th(\{\neg\varphi\})\big) \not\vdash_{D,Ske} \chi$.

Subsection 6.5

Inference relations based on preferential models

On preferential models

The idea of preferential models is to generalize the concept of minimal models that were used for circumscription.

Reasoning under preferential models semantics does no longer consider all models to compute consequences, but only preferred ones.

In the following we abstract from the underlying logic and use $\mathcal{L}$ and its elements ('propositions').

Definition 6.10 (preferential model structure)

A preferential model structure is a triple $(MS, \models, <)$, where

- MS is a set of models.
- $\models\ \subseteq MS \times \mathcal{L}$ is a relation between models and formulas in $\mathcal{L}$ and is called satisfaction relation of the structure.
- $<\ \subseteq MS \times MS$ is a relation on MS and is called the preference relation.

Let $m \in MS$ and $L \subseteq \mathcal{L}$.
Model m preferentially satisfies L, denoted by $m \models_{<}$ iff
$m \models$ and there is no model $m' \in MS$ s.t. $m' < m$ and $m' \models$.
We call m a preferential model of .

The intuition for the

- satisfaction relation $\models$ is that it states which formulas are satisfied by which models.
- preference relation $<$ is that it states which models are preferred over which other model.

Definition 6.11 ($\models_<$, $C_<$)

Based on preferential models we define an inference relation $\mathrel{|}\!\sim_<$ determined by a preferential model structure $(MS, \models, <)$ as:

$$L \mathrel{|}\!\sim_< x \text{ iff for all } m \in MS, m \models_< L \text{ implies } m \models x.$$

The inference operation $C_<$ determined by a preferential model structure $(MS, \models, <)$ is defined as follows:

$$C_<(L) = \{x \in \mathcal{L} \mid \text{ for all } m \in MS, m \models_< L \text{ implies } m \models x\}.$$

Intuition:
Formula x follows nonmonotonically from a set of formulas L if it is satisfied by all preferential models of L.

Theorem 6.12

Every preferential model structure satisfies Inclusion, Idempotence and Cut.

Proof: blackboard

Pure conditions of preferential model structures

Preferential model structures do not satisfy Cumulativity.

Example 6.13

Let MS be the infinite set $\{m_1, m_2, \ldots\}$, and define $m_i < m_j$ iff $j < i$.
Let also x, y, z be elements of the underlying language $\mathcal{L}$ such that:

$$m_i \models x \text{ for all } i > 0$$
$$m_i \not\models y \text{ for all } i > 0$$
$$m_i \models z \text{ iff } i = 1$$

There is no minimal model satisfying x, therefore $\{x\} \mid\!\sim\; y$ and $\{x\} \mid\!\sim\; z$.
But m_1 is the minimal model satisfying $\{x, z\}$ and $m_1 \not\models y$.
So, $\{x, z\} \mid\!\sim\; y$ is false and Cautious Monotony and thus Cumulativity is violated.