Completeness of circumscription

Completeness of circumscription need **not** hold

\( T \cup \{ \text{Circum}(T, P) \} \) is too weak, since it admits too many models. The relations in the models are expressible in FOL, but there exist relations (on the interpretation domain) not expressible in FOL.

In the circumscription schema \( \text{Circum}(T, P) \) describes \( P \)-minimality only for relations expressible in predicate logic.

**How to re-gain completeness?**

Use **second order logic** to quantify over relations.

Then the circumscription schema becomes:

\[
\forall \psi \left( \varphi[p/\psi] \land \forall X_1, \ldots, X_n (\psi \rightarrow p(X_1, \ldots, X_n)) \right) \\
\rightarrow \left( \forall X_1, \ldots, X_n (p(X_1, \ldots, X_n) \rightarrow \psi) \right).
\]

Now \( \psi \) is **not** limited to predicate expressions anymore!
Subsection 5.4

Consistency and expressive power
Consistency preservation

Consistency preservation:
if \( T \) is consistent, then \( T \cup \{\text{Circum}(T, P)\} \) consistent as well.

Predicate circumscription does not preserve consistency.

A special case:
A set of closed formulas is called universal iff the prenex normal form of all of its formulas does not contain any existential quantifier.

Theorem 5.7
Let \( T \) be a finite, consistent, universal set of closed formulas, and \( P \) a finite set of predicate symbols.
Then there exists a \( P \)-minimal model of \( T \). Consequently, \( T \cup \text{Circum}(T, P) \) is consistent.
Expressive power of circumscription

By (predicate) circumscription no new facts regarding the predicates not being circumscribed can be derived about ground terms.

**Theorem 5.8**

Let $T$ be a finite, universal set of closed formulas, $P$ a finite set of predicate symbols, $p$ an $n$-ary predicate symbol with $p \notin P$ and $t_1, \ldots, t_n$ ground terms. Then

1. $T \vdash_{\text{Circ}(P)} p(t_1, \ldots, t_n)$ iff $T \models p(t_1, \ldots, t_n)$.
2. $T \vdash_{\text{Circ}(P)} \neg p(t_1, \ldots, t_n)$ iff $T \models \neg p(t_1, \ldots, t_n)$.

Proof: blackboard
Applying Theorem 5.8 to the Tweety example

Consider again:
\[ \forall X (\text{bird}(X) \land \neg \text{abnormal}(X) \rightarrow \text{flies}(X)) \]
\[ \text{bird}(\text{tweety}) \]

By Theorem 5.8 the circumscription of abnormal cannot derive \text{flies}(\text{tweety})!

To see this, consider the interpretation \( I \) with

- \( \text{dom}(I) = \{1\} \),
- \( \text{tweety}^I = 1 \)
- \( \text{bird}^I = \text{abnormal}^I = \{1\} \), and
- \( \text{flies}^I = \emptyset \)

Now, \( I \) is a model of \( T \) and \text{flies}(\text{tweety}) \) is not true in \( I \).

\( I \) is \{abnormal\}-minimal, since \( \text{abnormal}^I \) cannot be reduced while keeping \( \text{flies}^I \) and validity of \( T \). Then by Theorem 5.6 it follows that \( T \not\models_{\text{Circ}(\text{abnormal})} \text{flies}(\text{tweety}) \).

Predicate circumscription does not suffice to realize default reasoning!
Section 6

Nonmonotonic Inference Relations

Subsection 6.1

Inference relations
Introduction

We have discussed three formalisms that provide nonmonotonic reasoning: Default logics, Autoepistemic logic, and Circumscription. They provide useful inference relations.

In this chapter we take a more abstract view:

- **What are the properties of nonmonotonic inference relations in general—独立于特定形式化？**
- **How to compare the different approaches?**

Inferences in the discussed formalisms:

- **Circumscription** uses minimal, i.e., preferred models, yielding a preferential inference relation.
- **Default logic** and **autoepistemic logic** use some kind of fixpoint construction.

What is the semantics of their inference relation?
Inference relations

Given knowledge about the world, when could a formula $\varphi$ be reasonably concluded from a set of formulas $M$?

Let $T$ be a set of first order sentences (the agent’s knowledge).

- **classical entailment**
  The agent classically infers a formula $\varphi$, if $\varphi$ holds in all models of $T$ in which $M$ holds.

But an agent’s knowledge might be incomplete, so nonmonotonic (/defeasible) inference relation are interesting.

- **under circumscription**, the agent might infer $\varphi$ from $M$ if $\varphi$ holds in every minimal model of $T$

- **in default logic**
  (if the agent’s knowledge is given also by a set of default rules $D$), the agent might infer $\varphi$ from $M$, if $\varphi$ is in every extension of the default theory $(D, \{M\})$.

Inference relations can be modeled as binary relations on (sets of formulas of) a logic $\mathcal{L}$. We denote nonmonotonic inference relations by $\not\mathrel{\vdash}$.

Which binary relations on $\mathcal{L}$ are (non-monotonic) inference relations?
For example: inference relation for default logic

In default logic the inference relation could be defined as:
Given a countable set of defaults $D$.

\[ W \models \varphi \iff \varphi \text{ is included in all extensions of } (W, D). \]

or alternatively

\[ W \models \varphi \iff \varphi \text{ is included in some extension of } (W, D). \]
The setting considered

In this chapter we consider:

- propositional logic
- inference relation $\vdash$
- the inference operation $C$ defined (for a given set of formulas $M$) as:

$$C(M) = \{\varphi \mid M \vdash \varphi\}.$$ 

We will use the inference relation $\vdash$ and the inference operation $C(M)$ interchangeably. E.g.: $M \vdash \varphi$ if $\varphi \in M$ can be formulated as $M \subseteq C(M)$.

The properties discussed in the following also hold for the classical inference relation $\vdash$.
(But not all properties of $\vdash$ hold for nonmonotonic reasoning.)
Subsection 6.2
Basic properties: pure conditions
Pure conditions

Consider the following properties of an inference operation $C$:

- $M \subseteq C(M)$ \hspace{1cm} \text{Inclusion}
- $C(M) = C(C(M))$ \hspace{1cm} \text{Idempotence}
- $M \subseteq N \subseteq C(M)$ implies $C(N) \subseteq C(M)$ \hspace{1cm} \text{Cut}
- $M \subseteq N \subseteq C(M)$ implies $C(M) \subseteq C(N)$ \hspace{1cm} \text{Cautious Monotony}
- $M \subseteq N \subseteq C(M)$ implies $C(M) = C(N)$ \hspace{1cm} \text{Cumulativity}
- $M \subseteq N$ implies $C(M) \subseteq C(N)$ \hspace{1cm} \text{Monotony}

These are called pure conditions, since they do not refer to any features of the underlying logic.
Intuition of the pure conditions

- **Inclusion**
  requires that the inference operation extends the set of formulas

- **Idempotence**
  requires that, after having applied the inference operation, another application does not add new formulas.

- **Cut**
  ensures that if the information in $M$ is expanded by some proposition included in the closure $C(M)$, then no new conclusions are obtained.

- **Cautious Monotony** (converse of Cut)
  the addition of a lemma does not decrease the set of conclusions.

- **Cumulativity** (Cut and Cautious Monotony combined)
  ensures that lemmas can be safely used without affecting the supported conclusions.

- **Monotony**
  an extended set of premises gives an extended set of conclusions.

An inference relation that satisfies Inclusion, Idempotence, Cut, Cautious Monotony and Cumulativity is called a **cumulative inference relation**.
What is the minimal set of conditions for an inference relation to be cumulative?

**Theorem 6.1**

1. *Cut and Inclusion imply Idempotence.*
2. *Cautious monotony and Idempotence imply Cut.*

**Proof:** blackboard

Since Cumulativity implies Cautious Monotony and also Cut, we can conclude that Cumulativity and Inclusion are sufficient to obtain a cumulative inference relation.
Subsection 6.3
Basic properties: interaction with logical connectives
Properties linking classical and nonmonotonic logic

In the following we refer to propositional logic.

- \( Th(M) \subseteq C(M) \)  
  Supraclassicality
- \( Th(C(M)) = C(M) \)  
  Left Absorption
- \( C(Th(M)) = C(M) \)  
  Right Absorption
- \( Th(C(M)) = C(M) = C(Th(M)) \)  
  Full Absorption
Absorption and pure properties

Theorem 6.2

Let $C$ be a supra-logical inference relation.

1. If $C$ satisfies Idempotence, then it satisfies Left Absorption.
2. If $C$ is Cumulative, then it satisfies Full Absorption.

Proof: blackboard