Towards extensions – unwanted effects

1. "Ungrounded" beliefs
An extension must not contain "ungrounded" beliefs, i.e. every formula in the extension must be derivable from $W$ and the consequents of applied defaults. We require extensions to be minimal w.r.t. to these properties.

Consider: $T = (W, D)$ with $W = \{\text{german}\}$ and $D = \{ \frac{\text{german} : \text{drinksBeer}}{\text{drinksBeer}} \}$

Now, $E = Th(\{\text{german}, \neg\text{drinksBeer}\})$ is minimal w.r.t. to the properties, but unintuitive.

2. Applications of defaults can contradict the application of an earlier default.
Consider:

\[
\begin{align*}
\text{true} : \text{creditworthy} & \quad \text{true} : \neg\text{creditworthy} \\
\text{approveCredit} & \quad \neg\text{creditworthy}
\end{align*}
\]

We apply the first default, since nothing contradicts the assumption $\text{creditworthy}$. We then apply the second, since $\neg\text{creditworthy}$ is consistent with the knowledge, $\neg\text{creditworthy}$ is derived.

Inclusion of $\neg\text{creditworthy}$ shows a posteriori that, we should not have assumed $\text{creditworthy}$. 
Subsection 3.2
Operational semantics of Default Logic

- based on the process in which inferences are drawn
- gives a procedure that can be applied

Idea:
- apply defaults as long as possible
- If a default should not have been applied, backtrack and try an alternative
Operational Semantics

Given a default theory $T = (W, D)$ let $\Pi = (\delta_0, \delta_1 \ldots)$ be (a finite or infinite) sequence of defaults from $D$ without multiple occurrences. (Possible order in which some defaults from $D$ are applied.)

$\Pi[k]$ denotes the initial segment of sequence $\Pi$ of length $k$.  

Each sequence $\Pi$ is associated with two sets: $In(\Pi)$ and $Out(\Pi)$

- $In(\Pi) = Th(W \cup \{cons(\delta) \mid \delta \text{ occurs in } \Pi\})$.
- $Out(\Pi) = \{\neg \psi \mid \psi \in just(\delta) \text{ for some } \delta \text{ in } \Pi\}$.

Intuition:

- $In(\Pi)$ represents the current knowledge base after the defaults in $\Pi$ have been applied
- $Out(\Pi)$ represents the formulas that should not become true even after subsequent application of other defaults.
Example: default sequences

Example 3.3
Consider $T = (W, D)$ with $W = \{a\}$ and the defaults from $D$:

$$\delta_1 = \frac{a}{\neg b}, \quad \delta_2 = \frac{b}{c}$$

For $\Pi_a = (\delta_1)$ we have $\text{ln}(\Pi_a) = \text{Th}(\{a, \neg b\})$ and $\text{Out}(\Pi_a) = \{b\}$.
For $\Pi_b = (\delta_2, \delta_1)$ we have $\text{ln}(\Pi_b) = \text{Th}(\{a, c, \neg b\})$ and $\text{Out}(\Pi_b) = \{-c, b\}$

We have not assured that the defaults can be applied in the order given. $(\delta_2, \delta_1)$ cannot be applied in this order, since $b \not\in \text{ln}(\cdot) = \text{Th}(W) = \text{Th}(a)$.

"Applicable sequences" are formalized by the notion of a process.
Process

Definition 3.4 (Process, successful, closed)

II is a process of $T$ iff $\delta_k$ is applicable to $\text{In}(\Pi[k])$ for every $k$ s.t. $^4 \delta_k$ occurs in $\Pi$. Let $\Pi$ be a process. We define:

- $\Pi$ is successful iff $\text{In}(\Pi) \cap \text{Out}(\Pi) = \emptyset$. Otherwise, it is failed.
- $\Pi$ is closed iff every $\delta \in D$ that is applicable to $\text{In}(\Pi)$ already occurs in $\text{In}(\Pi)$

Intuition:

Success of a process captures that is was "okay" to have assumed the justifications of the applied defaults; no formula $\neg \psi \in \text{Out}(\Pi)$ is part of the current knowledge base, so it was consistent to assume $\psi$.

Closed processes correspond to the extension being closed under application of the defaults.

$^4$"such that"
Example: properties of processes

Consider the default theory $T = (W, D)$ with $W = \{a\}$ and $D$ containing

$$
\delta_1 = \frac{a : \neg b}{d}, \quad \delta_2 = \frac{true : c}{b}
$$

$\Pi_1 = (\delta_1)$
is successful, but not closed, since $\delta_2$ may be applied to $ln(\Pi_1) = Th(\{a, d\})$.

$\Pi_2 = (\delta_2, \delta_1)$
is closed, but not successful. Since both $ln(\Pi_2) = Th(a, b, d)$ and $Out(\Pi_2) = \{b, \neg c\}$ contain $b$.

$\Pi_2 = (\delta_2)$
is a closed and successful process of $T$. 

Definition 3.5 (Extension)
Let $T$ be a default theory. A set of formulas $E$ is an extension of $T$ iff there is some closed and successful process $\Pi$ s.t. $E = Th(In(\Pi))$.

This definition may be applied directly to concrete examples.
To find a successful process, it suffices to generate a process $\Pi$, test whether $In(\Pi) \cap Out(\Pi) = \emptyset$ holds. If not, then backtrack.

A (in)finite default theory is a default theory, where $D$ has (in)finitely many elements.
For finite default theories ensuring closure is conceptually easy: apply an applicable default that has not been applied yet, until no more a left.

How about closure of infinite default theories?
Closure of infinite theories

Lemma 3.6

An infinite process $\Pi$ of a default theory $T = (W, D)$ is closed iff each default in $D$ that is applicable to $\ln(\Pi[k])$, for infinitely many numbers $k$, is already contained in $\Pi$.

Proof: blackboard

A strategy that guarantees the closure of an infinite process $\Pi$ must take care that any default which from $k$ on, demands application, will eventually be applied. This is the fairness condition from concurrent programming.
A systematic view on closed and successful processes

The process of finding an closed and successful process can be represented by a kind of (search) tree.

**Definition 3.7 (Process tree)**

Let $T = (W, D)$ be a default theory. A process tree is tree $G = (V, E)$, s.t. all nodes $v \in V$ are labeled with two sets of formulas:

- an In-set $In(v)$ and
- an Out-set $Out(v)$.

The root of $G$ is labeled with $Th(W)$ as In-set and $\emptyset$ as Out-set.

The paths of a process are the paths in $G$ starting at the root. A node $v$ is expanded if $In(v) \cap Out(v) = \emptyset$, otherwise it is marked "failed" and is a leaf of the process tree.

If $v$ is expanded it possesses for each default $\delta = \frac{\phi : \psi_1, \ldots, \psi_n}{\chi}$ one successor node $w$ that

- does not appear on the path from the root node to $v$ and
- is applicable to $In(v)$.
- is connected to $v$ by a edge labeled with $\delta$.
- is labeled with $Th(In(v) \cup \{\chi\})$ and $Out(v) \cup \{\neg\psi_1, \ldots, \neg\psi_n\}$. 
Subsection 3.3
Original semantics of default logics

- original definition by Ray Reiter
- fixed point based, not constructive
Consistency w.r.t. to what?

When applying defaults we need to ensure consistency. But consistency w.r.t. to which theory?

We consider again Example 3.2:

\[ T = (W, D) \text{ with } W = \{ \text{green, ADACmember} \} \text{ and } D = \{ \delta_1, \delta_2 \}, \text{ where} \]

\[ \delta_1 = \frac{\text{green : } \neg \text{likesCars}}{\neg \text{likesCars}} \quad \text{and} \quad \delta_2 = \frac{\text{ADACmember : } \text{likesCars}}{\text{likesCars}} \]

Consistency w.r.t. to alone \( W \) is not enough.
Solution by Reiter: Use a theory that plays the role of a context or belief set. Check consistency against this context.

A formalization of this idea:
A default \( \delta = \varphi : \psi_1, \ldots, \psi_n \) is applicable to a deductively closed set of formulas \( F \) w.r.t. belief set \( E \) (the context) iff \( \varphi \in F \) and \( \neg \psi_1 \notin E, \ldots, \neg \psi_n \notin E \) (each \( \psi_i \) is consistent with \( E \)).
Note that the concept “\( \delta \) is applicable to \( E \)” is so far a special case where \( E = F \).
Which context to use?

Observation:
If a default has been applied to a belief set $E$, a formula has been derived and is part of the knowledge base. Therefore it should be believed, i.e. become an element of belief set $E$.
On the other hand, $E$ should contain only formulas that can be derived from the axioms in $W$ by default application.

Definition 3.8 (Closure under a set of defaults w.r.t. a belief set)
Let $D$ be a set of defaults and $F$ a deductively closed set of formulas $F$.
$F$ is closed under $D$ w.r.t. belief set $E$ iff, for every default $\delta \in D$ that is applicable to $F$ w.r.t. belief set $E$, its consequent $\chi$ is also contained in $E$.

Lemma 3.9
Let $E' \subseteq E$ and $F$ be a set of formulas closed under some set of defaults $D$ w.r.t. $E'$. Then $F$ is closed under $D$ w.r.t. $E$.

Proof: exercise
Extension — original semantics

Definition 3.10 ($\Lambda_T(E)$, extension)

Given $T = (W, D)$ and a set of formulas $E$. Let $\Lambda_T(E)$ be the smallest\(^5\) set of formulas that is

- closed under deduction, i.e. contains all conclusions
- closed under $D$ w.r.t. $E$.

$E$ is an extension of $T$, iff $E = \Lambda_T(E)$.

Intuition:

- $\Lambda_T(E)$ contains all formulas that are sanctioned by $T$ w.r.t. $E$.
- $E$ is an extension, iff by the use of $E$ as a belief set, exactly the formulas in $E$ will be obtained from default application.

Observe: one first needs to guess $E$ and then check whether the fixed-point equation is fulfilled.

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\(^5\) i.e. has smallest number of elements
Are the two definitions equivalent?

**Theorem 3.11**

*Let $T = (W, D)$ be a default theory. E is an extension of $T$ (according Definition 3.5) iff $E = \Lambda_T(E)$.***

**Proof:** blackboard