Subsection 4.4
Computing expansions of AE-theories
Considerations

To achieve nonmonotonic behavior w.r.t. AE-theories, formulas ("conjectures") can be added to the set of believes that need not be added.

What makes computing expansions difficult?

- nested occurrences of the $L$-operator
- infinitely many conjectures. How to compute all expansions?

How to remedy this?

- Nested occurrences of $L$-operator: concentrate on potential kernels of expansions (Theorem 4.10).
- by Coincidence Lemma: it suffices to consider beliefs or non-beliefs in formulas from $\text{sub}(T)$ to determine the expansions of $T$.
  Only those formulas with $L$-operator play a role in the interpretation of $T$. 
Overview of the computation procedure for expansions

Compute expansions of AE-theories by:

- partition $\text{sub}(T)$ into:
  - $E(+)$: set of beliefs
  - $E(-)$: set of non-beliefs

- Compute the corresponding kernel $E(0)$ of a potential expansion, using $T$, beliefs in $E(+) \text{ and non-beliefs in } E(-)$.

- Check whether the stable set determined by $E(0)$ is indeed an expansion.
Example – Expansions of AE-theories without \(L\)-nestings

Example 4.14

Let \(T = \{Lp \rightarrow p\}\).

Since \(Lp \rightarrow p\) is the only AE-formula occurring (at top-level) of \(T\), \(\text{sub}(T) = \{p\}\).

There are two partitions of \(\text{sub}(T) = \{p\}\).

<table>
<thead>
<tr>
<th>(E(+))</th>
<th>(E(-))</th>
<th>(E(0))</th>
<th>(E(+) \subseteq E(0))?</th>
<th>(E(-) \cap E(0) = \emptyset)?</th>
<th>expansion?</th>
</tr>
</thead>
<tbody>
<tr>
<td>{p}</td>
<td>\emptyset</td>
<td>\text{Th}{p}</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>\emptyset</td>
<td>{p}</td>
<td>\text{Th}\emptyset</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

- \(E(0)\): set of first order formula that follow from \(T\).
- condition \(E(+) \subseteq E(0)\):
  test whether everything that the agent believes in is in \(E(0)\).
- condition \(E(-) \cap E(0) = \emptyset\):
  ensures that \(E(0)\) does not include non-beliefs of the agent
Procedure for computing expansions for AE-theories without L-nestings

Compute expansions no L-nesting ($T$)

1. $\text{Expansions} := \emptyset$
2. for all partitions $E(\text{+})$ and $E(\text{-})$ of $\text{sub}(T)$ do
3. \hspace{1em} $E(0) := \{\varphi \in \text{For}_0 \mid T \cup \text{LE}(\text{+}) \cup \neg \text{LE}(\text{-}) \models \varphi\}$
4. \hspace{1em} if $E(\text{+}) \subseteq E$ AND $E(\text{-}) \cap E = \emptyset$ then
5. \hspace{2em} $\text{Expansions} := \text{Expansions} \cup \{E(0)\}$
6. \hspace{1em} end if
7. end for
8. return $\text{Expansions}$
Example 4.15

Let \( T = \{ Lp \rightarrow p, \neg L \neg Lp \} \), with \( \text{sub}(T) = \{ p, \neg Lp \} \).

Now the partitions of \( \text{sub}(T) \) are no longer first order formulas!

<table>
<thead>
<tr>
<th>( E(+) )</th>
<th>( E(-) )</th>
<th>( E(0) )</th>
<th>( E(+) \subseteq E? )</th>
<th>( E(-) \cap E = \emptyset? )</th>
<th>expansion?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { p, \neg Lp } )</td>
<td>( \emptyset )</td>
<td>( \text{For}_0 )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( { p } )</td>
<td>( { \neg Lp } )</td>
<td>( \text{Th}({ p }) )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( { \neg Lp } )</td>
<td>( { p } )</td>
<td>( \text{For}_0 )</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>( \emptyset )</td>
<td>( { p, \neg Lp } )</td>
<td>( \text{Th}(\emptyset) )</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

- Line 1: \( E(0) \) is inconsistent, since \( \neg L \neg Lp \) follows from \( LE(+) \), but \( \neg L \neg Lp \in T \).
- Line 2: \( T \cup LE(+) \cup \neg LE(-) = \{ Lp \rightarrow p, \neg L \neg Lp, Lp \} \), thus \( E(0) = \text{Th}(\{ p \}) \). Since \( p \in E \) and \( E \) is stable and consistent, we have \( Lp \in E \) and thus \( \neg L \notin E \).
- Line 3: \( T \cup LE(+) \cup \neg LE(-) \) contains both \( L \neg Lp \) and \( \neg L \neg Lp \), thus \( E(0) = \text{For}_0 \).
- Line 4: \( T \cup LE(+) \cup \neg LE(-) = \{ Lp \rightarrow p, \neg L \neg Lp, \neg Lp \} \). From \( p \notin E \) follows \( \neg Lp \in E \) and thus \( E(-) \cap E \neq \emptyset \).
Procedure for computing expansions for general AE-theories

Compute expansions \((T)\)

1: \( \text{Expansions} := \emptyset \)
2: \textbf{for all} partitions \(E(+)\) and \(E(-)\) of \(\text{sub}(T)\) \textbf{do}
3: \(E(0) := \{ \varphi \in \text{For}_0 \mid T \cup LE(+) \cup \neg LE(-) \models \varphi \} \)
4: \textbf{Let} \(E\) be the unique stable set with kernel \(E(0)\)
5: \textbf{if} \(E(+ \subseteq E\) AND \(E(-) \cap E = \emptyset\) \textbf{then}
6: \(\text{Expansions} := \text{Expansions} \cup \{E(0)\}\)
7: \textbf{end if}
8: \textbf{end for}
9: \textbf{return} \(\text{Expansions}\)
Towards the correctness proof

Lemma 4.16 (Preservation Lemma)
Let $E$ be a stable set and $T$ an AE-theory. If $E_0 = \{ \varphi \in \text{For}_0 \mid T \cup LE \cup \neg LE^C \models \varphi \}$, then $E = \{ \varphi \in \text{For} \mid T \cup LE \cup \neg LE^C \models \varphi \}$.

Proof: blackboard

Lemma 4.17 (Coincidence Lemma)
Let $T$ be an AE-theory. Consider sets of AE-formulas $E(+)$, $E(-)$, $F(+)$, and $F(-)$ with the following properties:

• $\text{sub}(T) \subseteq E(+) \cup E(-)$ and $E(+) \cap E(-) = \emptyset$ and $\text{sub}(T) \subseteq F(+) \cup F(-)$ and $F(+) \cap F(-) = \emptyset$

• $E(+) \cap \text{sub}(T) = F(+) \cap \text{sub}(T)$

• $E(-) \cap \text{sub}(T) = F(-) \cap \text{sub}(T)$.

Then the same first order formula follow from $T \cup LE(+) \cup \neg LE(-)$ as from $T \cup LF(+) \cup \neg LF(-)$

Proof: blackboard
Correctness proof

**Theorem 4.18**

Let $T$ be an AE-theory and let $\text{sub}(T)$ be partitioned into the disjoint sets $E(\text{+})$ and $E(\text{-})$. We consider the following steps:

1. Compute $E_0 = \{ \varphi \in \text{For}_0 \mid T \cup LE(\text{+}) \cup \neg LE(\text{-}) \models \varphi \}$ and let $E$ be the unique stable set with kernel $E_0$.

2. Check whether $E(\text{+}) \subseteq E$ and $E(\text{-}) \cap E = \emptyset$.

Then the following holds:

a) If the check in Step 2. is positive, then $E$ is an expansion of $T$.

b) Conversely, for every expansion $E$ of $T$ there is a decomposition of $\text{sub}(T)$ into $E(\text{+})$ and $E(\text{-})$ such that

   - $E(0) = E_0$ and
   - the check in Step 2 is positive.

**Proof:** blackboard