Nominal Commutative Narrowing
(Work in progress)*

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Abstract

Narrowing equational problems is a well-known technique that adds to rewriting the required power to search for solutions of equational problems. Both rewriting and narrowing techniques are well-studied in first-order languages, but there is still a lot to investigate when dealing with languages with binders, such as nominal language. In a previous paper, by the second author, the nominal narrowing relation was introduced. In this abstract, we present a work in progress on the development of nominal commutative narrowing as a technique to solve nominal commutative unification problems. We have extended the definitions of nominal rewriting and nominal narrowing to take into account equational theories, and we are one step away from proving the Lifting Theorem relating nominal commutative narrowing and rewriting. The goal of this abstract is to present our ongoing research and its challenges as well as to obtain feedback from the community.

1 Introduction

The nominal framework [10] emerged as a promising approach to deal with languages with binders, such as lambda calculus, first-order logic, etc. In the nominal setting, equality coincides with the α-equivalence relation, and freshness constraints are part of the nominal reasoning, and not deemed to the meta-language. For example, we can express within the nominal language, as $a\#M$, the fact that if a name a occurs in a term M, it must be abstracted (in other words, a is fresh for M). To reason within this language, nominal unification [13] was developed.

Extensions of nominal unification with equational theories are being investigated. On the one hand, when an equational theory E can be presented by a convergent nominal rewrite system, nominal E-unification via nominal narrowing was already investigated [6]. On the other hand, when such presentation by convergent rewriting system does not exist, different extensions for dealing with rewriting modulo E were necessary. Initially, it was necessary to define the extensions of nominal equality with the theories associativity (A), commutativity (C) and associativity-commutativity (AC) [4]. Only recently, algorithms to solve nominal C-unification problems (and their formalisations) were proposed [1, 2, 7, 5].

In this abstract, we are interested in giving another step towards a more general development nominal E, treating the case in which E cannot be oriented as a convergent nominal rewrite system, thus extending the works [11, 14, 8]. We will report on our work in progress developing the nominal C-narrowing and C-rewriting relations, as well as present some of the extensions of the nominal language that were necessary to prove the nominal version of the Lifting Theorem (Corollary 3.5), that relates both narrowing and rewriting. The Lifting Theorem is fundamental

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to prove that nominal narrowing provides a sound and complete procedure for nominal C-unification. The theorem is partially proven, we are one-step away from completing the proof due to some extensions that still need to be done in the nominal language and properties that need to be verified.

2 Nominal Rewriting and Narrowing modulo E

In this section, we introduce our novel definitions of *equational nominal rewriting systems* (ENRS) and *equational nominal narrowing*, sometimes abbreviated to nominal E-rewriting systems and nominal E-narrowing. While we assume the reader's familiarity with nominal techniques, we briefly recap some basic definitions. For more details, we refer to [9].

Background. Fix countable infinite pairwise disjoint sets of *atoms* $\mathbb{A} = \{a, b, c, \dots\}$ and *variables* $\mathcal{X} = \{X, Y, Z, \dots\}$. Let Σ be a finite set of function symbols disjoint from $\mathbb{A}$ and $\mathcal{X}$ such that for each $f \in \Sigma$, a unique non-negative integer n (arity of f) is assigned. A *permutation* π is a bijection on $\mathbb{A}$ with finite domain, i.e., the set $\mathtt{dom}(\pi) := \{a \in \mathbb{A} \mid \pi(a) \neq a\}$ is finite. Nominal terms are defined inductively by the grammar: $s, t, u ::= a \mid \pi \cdot X \mid [a]t \mid f(t_1, \dots, t_n)$, where a is an *atom*, $\pi \cdot X$ is a moderated variable, $[a]t$ is the *abstraction* of a in the term t, and $f(t_1, \dots, t_n)$ is a *function application* with $f \in \Sigma$ and $f : n$. A term is *ground* if it does not contain (moderated) variables. A substitution is a mapping from variables (from $\mathcal{X}$) to (nominal) terms. Note that atoms are objects that can be bound and moderated variables are objects that can be instantiated by a substitution.

There are two kinds of constraints: $s \approx_\alpha t$ is an equality constraint; and $a\#t$ is a freshness constraint which means that a cannot occur unabstracted in t. Primitive constraints have the form $a\#X$ and ∇, Δ denote finite sets of primitive constraints. Judgements have the form $\Delta \vdash s \approx_\alpha t$ and $\Delta \vdash a\#t$ and are derived using the rules in Figure 1. In Figure 1 we use the *difference set* of two permutations $ds(\pi, \pi') := \{n \mid \pi \cdot n \neq \pi' \cdot n\}$. So $ds(\pi, \pi')\#X$ represents the set of constraints $\{n\#X \mid n \in ds(\pi, \pi')\}$. For example, if $\pi = (a\ b)(c\ d)$ and $\pi' = (c\ b)$, then $ds(\pi, \pi') = \{a, b, c, d\}$, and $ds(\pi, \pi')\#X = \{a\#X, b\#X, c\#X, d\#X\}$.

A term in context $\Delta \vdash t$ expresses that the term t has the freshness constraints imposed by Δ. For example, $a\#X \vdash f(X, h(b))$ expresses that a cannot occur free in instances of X when instantiating the term $f(X, h(b))$. Nominal rewriting rules can be defined under freshness constraints, i.e., $\nabla \vdash l \to r$.

Nominal rewriting relation $\to_{\mathsf{R}}$ is as expected:

$$\Delta \vdash s \to_{\mathsf{R}} t \iff s = \mathbb{C}[s'] \wedge \Delta \vdash \nabla\theta \wedge s' \approx_\alpha \pi \cdot (l\theta) \wedge t \approx_\alpha \mathbb{C}[\pi \cdot (r\theta)],$$

for a substitution θ, a subterm s' of s, a position $\mathbb{C}$ and a nominal rule $\nabla \vdash l \to r$.

Nominal E-rewriting and E-narrowing. Recall that an *equational term rewriting system* (ETRS), denoted $\mathsf{R} \cup \mathsf{E}$, is a set consisting of a theory T containing a set of axioms that can be split into a set R of rules and a set E of identities. To define (ETRS) we need to define the extended relation $\approx_{\alpha,\mathsf{E}}$, which takes into account α-equivalence and equality modulo E.

Remark 2.1. To define $\approx_{\alpha,\mathsf{E}}$ we need to extend the rules of Figure 1 with the dedicated rules for the identities defining E. For example, in the case of the theory C, we need to add the following rule

$$\frac{\Delta \vdash s_0 \ \approx_{\alpha,\mathsf{C}}\ t_i \qquad \Delta \vdash s_1 \ \approx_{\alpha,\mathsf{C}}\ t_{1-i} \qquad i = 0, 1}{\Delta \vdash f^{\mathsf{C}}(s_0, s_1) \ \approx_{\alpha,\mathsf{C}}\ f^{\mathsf{C}}(t_0, t_1)}\,(\approx_{\alpha,\mathsf{C}}\ \mathrm{C})$$

$$\frac{}{\Delta \vdash a\#b}(\#\text{atom}) \qquad \frac{(\pi^{-1}\cdot a\#X)\in\Delta}{\Delta \vdash a\#\pi\cdot X}(\#\text{var}) \qquad \frac{}{\Delta \vdash a\#[a]t}(\#a[a]) \qquad \frac{\Delta \vdash a\#t}{\Delta \vdash a\#[b]t}(\#a[b])$$

$$\frac{\Delta \vdash a\#t_1 \ \cdots \ \Delta \vdash a\#t_n}{\Delta \vdash a\#f(t_1,\cdots,t_n)}(\#\text{app}) \qquad \frac{}{\Delta \vdash a\approx_\alpha a}(\approx_\alpha \text{atom}) \qquad \frac{ds(\pi,\pi')\#X\in\Delta}{\Delta \vdash \pi\cdot X\approx_\alpha \pi'\cdot X}(\approx_\alpha \text{var})$$

$$\frac{\Delta \vdash s_1 \ \approx_\alpha \ t_1 \ \cdots \ \Delta \vdash s_n \ \approx_\alpha \ t_n}{\Delta \vdash f(s_1,\cdots,s_n) \ \approx_\alpha \ f(t_1,\cdots,t_n)}(\approx_\alpha \text{app}) \qquad \frac{\Delta \vdash s \ \approx_\alpha \ t}{\Delta \vdash [a]s \ \approx_\alpha \ [a]t}(\approx_\alpha [aa])$$

$$\frac{\Delta \vdash s \ \approx_\alpha \ (a\ b)\cdot t \qquad \Delta \vdash a\#t}{\Delta \vdash [a]s \ \approx_\alpha \ [b]t}(\approx_\alpha [ab])$$

Figure 1: Rules for $\#$ and $\approx_\alpha$

where f^{C} denotes that f^{C} is a commutative function symbol. Rule ($\approx_\alpha$ app) only applies when the function symbol f is not commutative. In addition, we need to modify the rules in Figure 1 to use $\approx_{\alpha,\mathsf{C}}$ instead of $\approx_\alpha$.

Extending the definition of ETRS to nominal terms with respect to a theory E, we obtain the following definition. Below, $[t]_{\approx_\mathsf{E}}$, denotes the equivalence class of the nominal term t modulo E, i.e., $[t]_{\approx_\mathsf{E}} = \{t' \mid t' \approx_{\alpha,\mathsf{E}} t\}$.

Definition 2.2 (Equational nominal rewrite system)**.** Let E be set of identities and R a set of nominal rewrite rules. A nominal term-in-context $\Delta \vdash s$, reduces with respect to R/E, when its equivalence class modulo E reduces via $\to_{\mathsf{R}/\mathsf{E}}$ as below.

$$\Delta \vdash ([s]_{\approx_\mathsf{E}} \to_{\mathsf{R}/\mathsf{E}} [t]_{\approx_\mathsf{E}}) \text{ iff there exist } s', t' \text{ such that } \Delta \vdash (s \approx_{\alpha,\mathsf{E}} s' \to_\mathsf{R} t' \approx_{\alpha,\mathsf{E}} t).$$

That said, we call R/E an *equational nominal rewrite system* (ENRS). In particular, R/C is a commutative nominal rewrite system.

Here we are dealing with α,E-congruence classes and they are in general infinite due to the availability of names for α-renaming. Although the pure $\approx_\alpha$ relation is decidable, when $\approx_\alpha$ is put together with an equational theory E which contains infinite congruence classes, the relation $\to_{\mathsf{R}/\mathsf{E}}$ may not be decidable (as in first-order). We will define the nominal relation $\to_{\mathsf{R},\mathsf{E}}$ that deals with nominal E-matching instead of inspecting the whole α,E-congruence class of a term.

Definition 2.3 (Nominal E-rewriting)**.** The *one-step* E*-rewrite relation* $\Delta \vdash s \to_{\mathsf{R},\mathsf{E}} t$ is the least relation such that for any $R = (\nabla \vdash l \to r) \in \mathsf{R}$, position $\mathbb{C}$, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathbb{C}[s'] \qquad \Delta \vdash \big(\nabla\theta,\ s' \approx_{\alpha,\mathsf{E}} \pi\cdot(l\theta),\ \mathbb{C}[\pi\cdot(r\theta)] \approx_{\alpha,\mathsf{E}} t\big)}{\Delta \vdash s \to_{\mathsf{R},\mathsf{E}} \ t}$$

The E-*rewrite relation* $\Delta \vdash s \to^*_{\mathsf{R},\mathsf{E}} t$ is the least relation that includes $\to_{\mathsf{R},\mathsf{E}}$ and satisfies: (i) for all Δ, s, s' we have $\Delta \vdash s \to^*_{\mathsf{R},\mathsf{E}} s'$ if $\Delta \vdash s \approx_{\alpha,\mathsf{E}} s'$; (ii) for all Δ, s, t, u we have that $\Delta \vdash s \to^*_{\mathsf{R},\mathsf{E}} t$ and $\Delta \vdash t \to^*_{\mathsf{R},\mathsf{E}} u$ implies $\Delta \vdash s \to^*_{\mathsf{R},\mathsf{E}} u$. If $\Delta \vdash s \to^*_{\mathsf{R},\mathsf{E}} t$ and $\Delta \vdash s \to^*_{\mathsf{R},\mathsf{E}} t'$, then we say that R is E-*confluent* when there exists a term u such that $\Delta \vdash t \to^*_{\mathsf{R},\mathsf{E}} u$ and $\Delta \vdash t' \to^*_{\mathsf{R},\mathsf{E}} u$. Also, R is said to be E-*terminating* if there is no infinite $\to_{\mathsf{R},\mathsf{E}}$ sequence. An ENRS R is called E-*convergent* if it is E-confluent and E-terminating. A term t is said to be in R,E-normal form (R/E-normal form) whenever one cannot apply another step of $\to_{\mathsf{R},\mathsf{E}}$ ($\to_{\mathsf{R}/\mathsf{E}}$).

In first-order languages, it is known that R,E-reducibility is decidable if the E-matching is decidable. Following Jouannaud et. al. [11], the existence of a finite and complete E-unification algorithm is a sufficient condition for that decidability. However solving nominal E-unification problems has the additional complication of dealing with renaming and freshness conditions, and these have a significant impact in obtaining finite and complete set of nominal E-unification algorithms.

Remark 2.4. Nominal C-unification is not finitary when one uses freshness constraints and substitutions for representing solutions [2], but the type of problems that generate an infinite set of C-unifiers are fixed-point equations $\pi \cdot X {}_?\overset{\mathsf{C}}{\approx}_? X$. For e.g., the nominal C-unification problem $(a\ b) \cdot X {}_?\overset{\mathsf{C}}{\approx}_? X$ has solutions $[X \mapsto a \oplus b], [X \mapsto (a \oplus b) \oplus (a \oplus b)], \ldots$. However, these problems do not appear in nominal C-matching [3]. Thus, the relation $\to_{\mathsf{R},\mathsf{C}}$ is decidable.

Now we define the nominal narrowing relation modulo E, extending previous works [6].

Definition 2.5 (Nominal E-narrowing)**.** The *one-step* E*-narrowing relation* $(\Delta \vdash s) \rightsquigarrow_{\mathsf{R},\mathsf{E}} (\Delta' \vdash t)$ is the least relation such that for any $R = (\nabla \vdash l \to r) \in \mathsf{R}$, position $\mathbb{C}$, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathbb{C}[s'] \qquad \Delta' \vdash \big(\nabla\theta,\ \Delta\theta,\ s'\theta \approx_{\alpha,\mathsf{E}} \pi \cdot (l\theta),\ (\mathbb{C}[\pi \cdot r])\theta \approx_{\alpha,\mathsf{E}} t\big)}{(\Delta \vdash s) \rightsquigarrow_{\mathsf{R},\mathsf{E}} (\Delta' \vdash t)}\ .$$

The *nominal* E*-narrowing relation* $(\Delta \vdash s) \rightsquigarrow^*_{\mathsf{R},\mathsf{E}} (\Delta' \vdash t)$ is the least relation that includes $\rightsquigarrow_{\mathsf{R},\mathsf{E}}$ and satisfies: (i) for all Δ, s, s' we have $(\Delta \vdash s) \rightsquigarrow^*_{\mathsf{R},\mathsf{E}} (\Delta \vdash s')$ if $\Delta \vdash s \approx_{\alpha,\mathsf{E}} s'$; (ii) for all $\Delta, \Delta', \Delta'', s, t, u$ we have that $(\Delta \vdash s) \rightsquigarrow^*_{\mathsf{R},\mathsf{E}} (\Delta' \vdash t)$ and $(\Delta' \vdash t) \rightsquigarrow^*_{\mathsf{R},\mathsf{E}} (\Delta'' \vdash u)$ implies $(\Delta \vdash s) \rightsquigarrow^*_{\mathsf{R},\mathsf{E}} (\Delta'' \vdash u)$.

The permutation π and substitution θ above are found by solving the E-unification problem $(\nabla \vdash l)\ {}_?\overset{\mathsf{E}}{\approx}_?\ (\Delta \vdash s')$. In this work, we will focus on the theory C, for which a nominal unification algorithm exists. From now on, our results are concentrated on $\to_{\mathsf{R}/\mathsf{C}}, \to_{\mathsf{R},\mathsf{C}}$ and $\rightsquigarrow_{\mathsf{R},\mathsf{C}}$.

Since nominal C-narrowing is defined on nominal C-unification, which is not finitary when we use pairs (Δ', θ) of freshness contexts and substitutions for representing solutions, and following the Remark 2.4, we can conclude that our nominal C-narrowing trees are infinitely branching.

Example 1. Consider the signature $\Sigma = \{h : 1, f : 2, \oplus : 2\}$, where $f, \oplus$ are commutative symbols. Let $R = \{\ \vdash h(Y) \to h(Y),\ \vdash f([a][b] \cdot Z, Z) \to f(h(Z), h(Z))\}$ be a set of rewrite[1] rules and $\mathsf{C} = \{\ \vdash f(X,Y) \approx f(Y,X),\ \vdash X \oplus Y \approx Y \oplus X\}$ be the axioms defining the theory.

Let $\vdash h(f([b][a]X, X))$ be a nominal term that we want to apply nominal narrowing modulo C. Observe that we can apply one step of narrowing, and then we obtain a branch that yields infinite branches due to the fixed-point equation (see Figure 2).

3 Nominal Lifting Theorem modulo C

Let $\mathsf{R} = \{\nabla_i \vdash l_i \to r_i\}$ be a C-convergent NRS. We want to establish correspondence between nominal C-narrowing and nominal C-rewriting. We will do that via an extension of the Nominal Lifting Theorem (cf. Theorem 12 [6]) for the extended relations $\rightsquigarrow_{\mathsf{R},\mathsf{C}}$ and $\to_{\mathsf{R},\mathsf{C}}$.

We start by defining a normalised substitution with respect to the relation $\to_{\mathsf{R},\mathsf{C}}$:

[1] $\vdash l \to r$ denotes $\emptyset \vdash l \to r$.

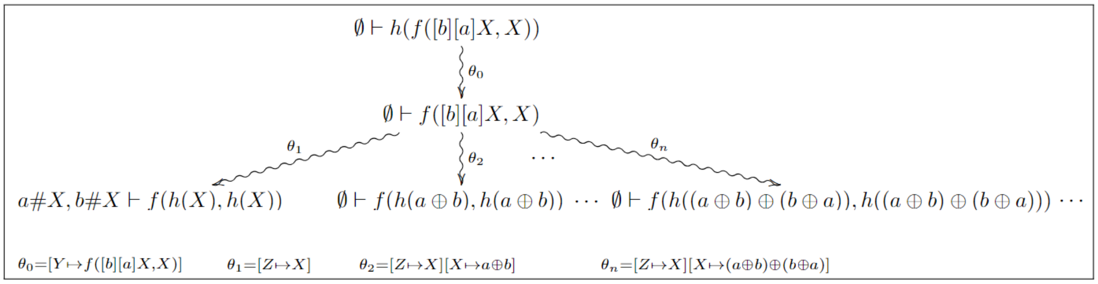


Figure 2: Infinitely branching tree

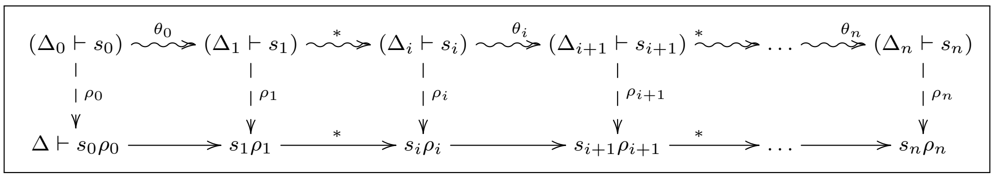


Figure 3: Corresponding Narrowing to Rewriting Derivations

Definition 3.1 (Normalised substitution w.r.t $\to_{\mathsf{R,C}}$)**.** A substitution θ is *normalised in* Δ *with relation to* $\to_{\mathsf{R,C}}$ if $\Delta \vdash X\theta$ is a $\mathsf{R,C}$-normal form in R for every X. A substitution θ satisfies the freshness context Δ if there exists a freshness context ∇ such that $\nabla \vdash a\#X\theta$ for each $a\#X \in \Delta$. The minimal such ∇ is $\langle \Delta\theta \rangle_{nf}$, the latter which denotes the normal form of the set of freshness constraints $\Delta\theta$, after bottom-up simplification using the rules in Figure 1.

The next result (correctness) states that for each finite sequence of narrowing steps, there exists a finite sequence of rewriting steps.

Theorem 3.2. *($\rightsquigarrow^*_{\mathsf{R,C}}$ to $\to^*_{\mathsf{R,C}}$) Let $(\Delta_0 \vdash s_0) \rightsquigarrow^*_{\mathsf{R,C}} (\Delta_n \vdash s_n)$ be a nominal* C*-narrowing derivation. Let ρ be a substitution satisfying Δ_0, i.e., there exists Δ such that $\Delta \vdash \Delta_0\rho$. Then, there exists a rewriting derivation*

$$\Delta \vdash s_0\rho_0 \to^*_{\mathsf{R,C}} s_n\rho$$

such that $\Delta \vdash \Delta_i\rho_{i+1}$ and $\rho_i = \theta_i \ldots \theta_{n-1}\rho$, for all $0 \leq i < n$. In other words,

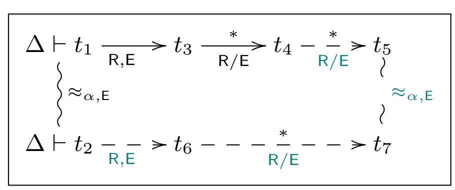
$$\Delta \vdash (s_0\theta)\rho \to^*_{\mathsf{R,C}} s_n\rho$$

where θ is the composition of the successive $\mathsf{R,C}$*-narrowing substitutions.*

Proof. By induction on the length of the narrowing derivation and illustrated in Figure 3. □

The proof of the converse (completeness) is more challenging. In the first-order case, the approach to prove that each finite sequence of $\to_{\mathsf{R,C}}$ steps corresponds to a finite sequence of $\rightsquigarrow_{\mathsf{R,C}}$ steps relies on an additional property: E-coherence. This property can be extended to the nominal framework as follows:

Definition 3.3 (Nominal E-Coherence)**.** The relation $\Delta \vdash \_ \to_{\mathsf{R,E}} \_$ is called E-*coherent* iff for all t_1, t_2, t_3, t_4 such that $\Delta \vdash t_1 \approx_{\alpha,\mathsf{E}} t_2$ and $\Delta \vdash t_1 \to_{\mathsf{R,E}} t_3 \to^*_{\mathsf{R/E}} t_4$, there exist t_5, t_6, t_7 such that $\Delta \vdash t_4 \to^*_{\mathsf{R/E}} t_5$, $t_2 \to_{\mathsf{R,E}} t_6 \to^*_{\mathsf{R/E}} t_7$ and $\Delta \vdash t_5 \approx_{\mathsf{E}} t_7$, for some Δ.



The diagram above illustrates nominal E-coherence: the dashed lines represent existentially quantified reductions.

Jouannaud et. al. [11, 12] proved, in the first-order case, that $\rightarrow_{\mathsf{R/E}}$ and $\rightarrow_{\mathsf{R,E}}$ coincide, if $\rightarrow_{\mathsf{R,E}}$ is E-coherent. We conjecture that the same result holds for the nominal framework, as long as E is a first-order theory (it does not contain bindings and/or freshness constraints).

Conjecture 3.1. *Let* E *be a first-order theory and* R *be a nominal rewrite system that is* E*-confluent and* E*-terminating. Then the* $\mathsf{R,E}$*- and* $\mathsf{R/E}$*-normal forms of any term* t *are* E*-equal iff* $\rightarrow_{\mathsf{R,E}}$ *is* E*-coherent.*

Proof. As in the first order case, we conjecture that the proof will follow by case analysis on the normal forms $t{\downarrow_{\mathsf{R,E}}}$ and $t{\downarrow_{\mathsf{R/E}}}$ of a term t. In addition, it will use the fact that $\rightarrow_{\mathsf{R,E}}\subseteq\rightarrow_{\mathsf{R/E}}$. □

Conjecture 3.1 is used in the proof of Theorem 3.4, which we call a *naive completeness*: the exact conditions on the freshness contexts have to be further investigated and Conjecture 3.1 has to be proven.

Theorem 3.4 (Naive version of Proposition 3 in [11])**.** *Let* $\mathsf{R}\cup\mathsf{C}$ *be an ENRS such that* R *is* C*-confluent and* C*-terminating and* $\rightarrow_{\mathsf{R,C}}$ *is* C*-coherent. Let* V_0 *be a finite set of variables containing* $V = V(\Delta_0, s_0)$*. Then, for any* $\mathsf{R,C}$*-derivation*

$$\Delta \vdash t_0 = s_0\rho_0 \rightarrow^*_{\mathsf{R,C}} t_0{\downarrow}$$

to any of its $\mathsf{R,C}$*-normal forms, say* $t_0{\downarrow}$*, where* $\mathit{dom}(\rho_0) \subseteq V(s_0) \subseteq V_0$ *and* ρ_0 *is a* $\mathsf{R,C}$*-normalised substitution that satisfies* Δ_0 *with* Δ*, there exist a* $\mathsf{R,C}$*-narrowing derivation*

$$(\Delta_0 \vdash s_0) \rightsquigarrow^*_{\mathsf{R,C}} (\Delta_n \vdash s_n),$$

for each i*,* $0 \leq i < n$*, with the composition of substitutions* θ*, and a* $\mathsf{R,C}$*-normalised substitution* ρ_n *such that* $\Delta \vdash s_n\rho_n \approx_{\alpha,\mathsf{C}} t_0{\downarrow}$ *and* $\Delta \vdash \rho_0|_V \approx_{\alpha,\mathsf{C}} \theta\rho_n|_V$*.*

Proof. By induction on the length of the rewriting derivation. □

Thus, we obtain our main result:

Corollary 3.5 (C-Lifting Theorem)**.** *Nominal lifting modulo* C *is a consequence of Theorem 3.2 and Theorem 3.4.*

The C-Lifting Theorem is fundamental to prove that nominal narrowing provides a sound and complete procedure for nominal C-unification.

4 Conclusion and Future Work

In this work, we proposed definitions for nominal $\mathsf{R,C}$-rewriting and $\mathsf{R,C}$-narrowing and proved some properties relating them. So far, we have partially proved the nominal version of the Lifting Theorem taking into account commutativity. The next step is to complete its proof. Since nominal C-unification based on freshness constraints only is not finitary, our nominal C-narrowing tree is not finite. As future work, we plan to investigate alternative approaches to nominal C-unification for which the representation of solutions is finite, such as the approach using fixed-point constraints.

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