TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Term Rewriting Systems

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Exercise Sheet 4 – Orders and Confluence

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Exercise 4.1 Consider the *Ackermann function*, which is defined as follows.

$$\begin{aligned} \mathsf{ack}\colon \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (n,m) &\mapsto \begin{cases} m+1 & \text{if } n = 0 \\ \mathsf{ack}(n-1,1) & \text{if } m = 0 \\ \mathsf{ack}(n-1,\mathsf{ack}(n,m-1)) & \text{otherwise} \end{cases} \end{aligned}$$

Show that ack is well-defined.

Hint. Choose an appropriate order on $\mathbb{N} \times \mathbb{N}$ and use well-founded induction to prove that the function ack is well-defined, i.e., for each pair (n,m) of natural numbers, the value $\mathsf{ack}(n,m)$ is uniquely determined by the formula above.

Exercise 4.2 Prove that, if $>_A$ is a linear strict order on A and $>_B$ is a linear strict order on B, then the lexicographic product $>_{A\times B}$ is a linear strict order on $A \times B$.

Exercise 4.3 Let $(A, \succ)$ be a strictly ordered set. Prove or refute the following claims.

(a) If A has a smallest element, then $\succ$ is well-founded.

(b) If every non-empty subset of A has a smallest element, then $\succ$ is well-founded.

(c) If $\succ$ is well-founded, then, for each element $a \in A$, there are only finitely many $b \in A$ with $a \succ b$.

(d) If $\succ$ is well-founded, then, for each element $a \in A$, there is an $n_a \in \mathbb{N}$ such that each $\succ$-path starting at a is of length at most n_a.

(e) If $\succ$ is well-founded, then each non-empty subset of A has a smallest element.

★ **Exercise 4.4** A reduction $\Rightarrow$ is a *refinement* of $\rightarrow$ if $\rightarrow \subseteq \overset{*}{\Rightarrow}$, and a *compatible refinement* if additionally $x \overset{*}{\Rightarrow} y$ implies $x \downarrow y$.

Show the following statements.

(a) Let $\Rightarrow$ be a refinement of $\rightarrow$. Then $\Rightarrow$ is a compatible refinement of $\rightarrow$ if, and only if, $x \Rightarrow y \overset{*}{\rightarrow} z$ implies $x \downarrow z$.

(b) Let $\Rightarrow$ be a compatible refinement of $\rightarrow$. Then $\Rightarrow$ is confluent if, and only if, $\rightarrow$ is confluent.

Hint. Use (a) for proving the only if direction.

Exercise 4.5 Prove that, if $\rightarrow$ enjoys the diamond property, then every element either is in normal form or does not have a normal form.

Hint. Use to the diamond property to show that each chain of reductions issuing from a reducible element can be extended by a further element.

Exercise 4.6 Give a counterexample which shows that strong confluence does not imply the following property.

$$y_1 \leftarrow x \rightarrow y_2 \quad \text{implies} \quad \exists z\colon y_1 \stackrel{=}{\rightarrow} z \stackrel{=}{\leftarrow} y_2$$

Exercise 4.7 Which of the following statements hold true? Justify your answer.

(a) If $x \downarrow y$ and $y \downarrow z$, then $x \downarrow z$.

(b) A relation is confluent if, and only if, it is strongly confluent.

(c) Every terminating relation is confluent.

(d) Every normalizing relation terminates.