Term Rewriting Systems

Summer Semester 2018

Exercise Sheet 5 – Universal Algebra

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Prof. Dr.-Ing. Franz Baader, Dipl.-Math. Francesco Kriegel

Exercise 5.1 Let $T(\Sigma, \{x\})$ be the set of Σ-terms over one variable x. The reduction relation $\rightarrow_I$ on $T(\Sigma, \{x\})$ is defined by $s \rightarrow_I t$ if s is an instance of t and $s \neq t$. Show that $\rightarrow_I$ is terminating and confluent. Can this result be generalized to the case of more than one variable?

Exercise 5.2 Let E be a set of identities with some $\ell \approx r \in E$ such that either ℓ is a variable or $\mathsf{Var}(r) \not\subseteq \mathsf{Var}(\ell)$. Prove that $\rightarrow_E$ is not terminating.

Exercise 5.3 Let E be a set of Σ-identities. Show the following statements.

(a) The reduction relation $\rightarrow_E$ is closed under substitutions and compatible with Σ-operations.

(b) The relation $\stackrel{*}{\leftrightarrow}_E$ is closed under substitutions and compatible with Σ-operations.

Exercise 5.4 Let $\equiv$ be a binary relation on $T(\Sigma, V)$. Prove the following statements.

(a) The relation $\equiv$ is compatible with Σ-operations if, and only if, it is compatible with Σ-contexts.

(b) If $\equiv$ is reflexive and transitive, then it is compatible with Σ-operations if, and only if, it is closed under Σ-operations.

★ **Exercise 5.5** Let $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, where $\mathcal{K}$ is a class of Σ-algebras that contains a finite algebra of cardinality larger than 1, and assume that $\mathcal{A}$ is free in $\mathcal{K}$ with finite generating set X and that $\mathcal{B}$ is free in $\mathcal{K}$ with finite generating set Y. Show that the following holds: if $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then $|X| = |Y|$.

Hint. Let $\mathcal{F}$ be a finite algebra in $\mathcal{K}$ with at least two elements. Either show that the generator of $\mathcal{A}$ induces another generator of $\mathcal{B}$ or prove that there is a bijective mapping between homomorphisms from $\mathcal{A}$ to $\mathcal{F}$ and homomorphisms from $\mathcal{B}$ to $\mathcal{F}$, and then determine the respective number of homomorphisms $\mathcal{A} \rightarrow \mathcal{F}$ and $\mathcal{B} \rightarrow \mathcal{F}$.

Exercise 5.6 Let $\Sigma := \{f\}$ for a binary function symbol f, and

$$\mathsf{AC} := \{f(x, f(y, z)) \approx f(f(x, y), z),\ f(x, y) \approx f(y, x)\}.$$

Consider the Σ-algebra $\mathcal{N}$ that has $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \{(0, 0, 0)\}$ as carrier set, and which interprets f as component-wise addition. Show that $\mathcal{N}$ is the free algebra in $\mathcal{V}(\mathsf{AC})$ with a generating set of cardinality 3.